Stabilized Virtual Element Method for the nonlinear convection-diffusion problems

A thesis submitted

in partial fulfillment for the award of the degree of

Doctor of Philosophy

by

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Certificate

This is to certify that the thesis titled ***Stabilized Virtual Element Method for the nonlinear convection-diffusion problems*** submitted by **M.Arrutselvi**, to the Indian Institute of Space Science and Technology, Thiruvananthapuram, in partial fulfillment for the award of the degree of **Doctor of Philosophy** is a bona fide record of the original work carried out by him/her under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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Declaration

I declare that this thesis titled ***Stabilized Virtual Element Method for the nonlinear convection-diffusion problems*** submitted in partial fulfillment for the award of the degree of **Doctor of Philosophy** is a record of the original work carried out by me under the supervision of **Dr. E.Natarajan**, and has not formed the basis for the award of any degree, diploma, associateship, fellowship, or other titles in this or any other Institution or University of higher learning. In keeping with the ethical practice in reporting scientific information, due acknowledgments have been made wherever the findings of others have been cited.

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This thesis is dedicated to my family. I am everlastingly thankful to God for gifting me with such an extraordinary family.

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Abstract

This work concerns residual-based stabilization of the Virtual Element Method for non-linear convection-diffusion problems. It is well-known that the numerical simulations of singularly perturbed problem produce solutions with spurious oscillations. In chapter one, we discuss the Galerkin approximation of the convection-diffusion equation. From the investigation of a simple one-dimensional problem, it is revealed that there is an onset of unphysical oscillations in the Galerkin solution for dominant convection. From one perspective, very rigorous mesh refinement acts as a remedy. As this resolve is non-viable, we study residual-based stabilization methods that circumvent mesh fine-tuning. Then we introduce the polytopal Galerkin method called the Virtual Element Method. We clearly state the advantage of VEM over the existing polytopal methods and briefly give the construction of the VEM space. We demonstrate the usage of the polynomial projection operators Π_p^{∇}, Π_p^0 and $\mathbf{\Pi}_{p-1}^0$.

Chapter two is devoted to studying the SUPG stabilization of VEM for the semilinear convection-diffusion-reaction equation. We prove theoretical estimates involving the mesh size h and the polynomial order p. For analysis, we prove the existence of an interpolation operator onto VEM space with optimal approximation property with respect to both the parameters $h,\ p$ for L^2 norm and H^1 semi-norm. Under suitable choice of the SUPG parameter, the error estimate showing optimal order of convergence is derived. We obtain the optimal convergence rate in H^1 semi-norm and L^2 norm for convection-dominated and reaction-dominated phenomena, respectively. In fact we obtain optimal order for the energy norm $|||\cdot|||$. Numerical experiments conducted verified our theoretical results over convex and nonconvex meshes for VEM order $p = 1, 2, 3$.

The shock-capturing stabilization of VEM for the convection-diffusion equation is analyzed in chapter 3. We begin by formulating a computable VEM scheme stabilized with the shock-capturing technique for the linear convection-diffusion-reaction equation. It is noted that the discretization of a linear problem produced a nonlinear discrete scheme. The existence of the VEM solution was shown with the help of a variant of Brouwers fixed point theorem. The efficiency of the shock-capturing method was investigated numerically by comparing it with the SUPG method, for a linear problem with discontinuous boundary conditions, on different polygonal meshes. With the success of shock-capturing in reducing

spurious oscillations, we proceed to investigate in detail the shock-capturing stabilization of VEM for the semilinear convection-diffusion equation. We discussed two variants of shock-capturing technique, where in the first case, we add isotropic artificial diffusion, and the second type adds anisotropic diffusion. Error estimate with similar order of convergence as the SUPG method is derived. We used the Newton method in the simulations to solve a nonlinear system. Numerical experiments conducted reveal the effectiveness of the shock-capturing stabilization in diminishing the cross-wind oscillations present in the SUPG solution.

The fourth chapter discusses the SUPG stabilization of VEM for the quasilinear convection-diffusion-reaction equation. In this, we study the approximation of branches of nonsingular solutions. We show the existence and uniqueness of a branch of discrete solution approximating the branch of the nonsingular solution through results proved by Brezzi et al. for a much general class of nonlinear equations. Convergence estimate showing optimal order for H^1 seminorm and the energy norm $|||\cdot|||$ were derived. Since the problem is quasilinear, on the fine mesh using the Newton method to solve the system is time-consuming. Therefore we use the two-grid method that involves two meshes of different mesh sizes for solving the nonlinear system of equations. Numerical experiments conducted verified the theoretical results. The CPU time taken by the two-grid for solving the system is halved compared to the time taken by the Newton method on a fine mesh.

In Chapter 5, we consider the discretization of the nonlocal coupled parabolic problem within the framework of the virtual element method. In fully discrete formulation, the backward Euler method is used for discretizing the time derivative, and VEM is used for spatial discretization. The presence of nonlocal coefficients makes the computation of the Jacobian more expensive in Newton's method and destroys the sparsity of the Jacobian. In order to resolve this problem, we propose an equivalent formulation that yields a sparse Jacobian. We derive the error estimates in the L^2 and H^1 norms. A linearised scheme without compromising the convergence rate in different norms is proposed to reduce the computational complexity further. Finally, the theoretical results are verified through the numerical experiments conducted on arbitrary polygonal meshes.

The final chapter discusses the possible works related to problems studied in this thesis that can be investigated in the future.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Convection-diffusion-reaction equation

The convection-diffusion-reaction equation arise in several fields such as fluid dynamics. Some engineering applications include tracking contaminant spread in a moving water body, pollutant transport in atmosphere, water percolation and flow in porous media (water monitoring), pressure variation of the wind surrounding the wings of an aircraft, tracing oil spill space-time evolution in oceans and subsurface flow problems like crude oil extraction and gas storage beneath the sea bed or land surface. A few biological applications are blood flow in the arteries, and tissue physiology and morphogenesis depending on diffusion of chemical morphogens in the extra-cellular fluid or matrix.

On a bounded domain $\Omega \subset \mathbb{R}^2$, consider a simple steady linear convection-diffusion equation given by,

$$\begin{aligned} -\nabla \cdot (K \nabla u) + \mathbf{b} \cdot \nabla u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.1.1}$$

The equation in (1.1.1) involves combination of convection and diffusion processes. The solution $u(x)$ is the variable of interest such as species concentration for mass transfer or temperature concentration for heat transfer. The variable $K > 0$ is the diffusion coefficient such as mass diffusivity for particle motion or thermal diffusivity for heat transport. The function $\mathbf{b}(x) \in [L^\infty(\Omega)]^2$ is the velocity field with which the quantity u is moving. The right hand-side function f gives the source or sink of quantity u. The term $-\nabla \cdot (K \nabla u)$ describes diffusion and the term $\mathbf{b} \cdot \nabla u$ represent convection.

The weak form of (1.1.1) is, Find $u \in H_0^1(\Omega)$ such that

$$(K\,\nabla u,\, \nabla v)_\Omega + (\mathbf{b}\cdot\nabla u,\, v)_\Omega \;=\; (f,\, v)_\Omega \qquad \forall\, v \in H_0^1(\Omega). \tag{1.1.2}$$

The Galerkin (or numerical) approximation of (1.1.2) produces numerical solutions with unphysical oscillations. When the non-symmetric convection term dominates, the best approximation property in the energy norm of the Galerkin method is affected. This leads to poor approximation of the solution of weak form (1.1.2) by the numerical methods. We briefly investigate the Galerkin approximation of the convection-diffusion equation using a simple 1-D example (refer [1]) whose exact solution is known. Consider 1-D convection-diffusion problem

$$wu_x - \epsilon u_{xx} = 1, \quad x \in [0,1] \tag{1.1.3}$$

$$u(0) = 0 \quad \text{and} \quad u(1) = 0. \tag{1.1.4}$$

Let us denote $\gamma = \dfrac{w}{\epsilon}$. The exact solution of the model problem (1.1.3) is

$$u(x) = \frac{1}{w}\left(x - \frac{1 - exp(\gamma x)}{1 - exp\gamma}\right).$$

Let us apply the Galerkin finite element method method to obtain a approximate solution of (1.1.3). The weak formulation is defined as : To find u $\in H_0^1(0,1)$ such that

$$\int_0^1 (vwu_x + v_x\epsilon u_x)\,dx = \int_0^1 v\,dx \quad \forall\, v \in H_0^1(0,1).$$

As usual, discretise [0,1] using a uniform mesh of linear elements of size 'h', with nodes $x_1, x_2, ..., x_n$. Let V_h be the finite dimensional subspace of $H_0^1(0,1)$ consisting of continuous peicewise polynomial functions. For implementation, on a element $(x_i,\, x_{i+1})$, we use the following shape functions :

$$N_1(\xi) = \frac{1}{2}(1-\xi) \qquad N_2(\xi) = \frac{1}{2}(1+\xi)$$

where ξ is the normalised coordinate, $-1 \le \xi \le 1$. Then on evaluating the bilinear forms on an interval (x_j, x_{j+1}) , we obtain, the local convection matrix,

$$\frac{w}{2}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

and the local diffusion matrix,

$$\frac{\epsilon}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We note that the diffusion matrix is symmetric, where as the convection matrix is non-symmetric. Let us introduce mesh Peclet number,

$$Pe = \frac{wh}{2\epsilon}.$$

In problem (1.1.3), choose $w = 1$ and varying ϵ. We solve the above 1-D problem using the Galerkin finite element method(GFEM). The numerical approximation is computed with a mesh of 10 uniform elements (i.e $h = 1/10$) for various mesh Peclet numbers $Pe = 0.25, 0.4, 0.7, 1.0, 1.5, 5$. A comparison of the plot of the exact solution and the GFEM solution for different Pe is shown in figure 1.1. We note that the approximation deteriorates as Peclet number approaches 1 and there is onset of oscillations when mesh Peclet number is equal to 1 or greater than 1.

Alternatively, when convection dominates or the diffusion coefficient K is very small, the solution of the model problem (1.1.1) develop layers where the magnitude of the solution vary drastically within a thin region. Layers typically arise near a boundary, where the solution must adhere to a boundary condition and layers may also occur in the interior of the domain due to discontinuities in the coefficients. Then the onset of oscillation can also be seen as a mesh resolution problem in the standard numerical methods. If mesh size h is choosen to be smaller than diffusivity , then no oscillations occur. But such an extent of mesh refinement is computationally very expensive and the method becomes practically inapplicable. We need to look for a technique that helps to prevent the outbreak of oscillations without the need for mesh refinement. Such a remedy is called stabilization method.

In this thesis, we consider a stabilization strategy that adds weighted residual to the numerical method - the well-known Streamline upwind Petrov-Galerkin (SUPG) method introduced by Brooks and Hughes in [2]. The SUPG method adds artificial diffusion along the streamline direction. For further improvement we use a nonlinear modification of SUPG stabilization called shock-capturing method. Unlike SUPG method, the shock-capturing method introduced in [3] satisfies the discrete maximum principle. In the subsequent chapters we shall discuss these stabilization method for nonlinear convection-diffusion equation in the setting of the recently introduced Galerkin method - the Virtual Element Method (VEM).

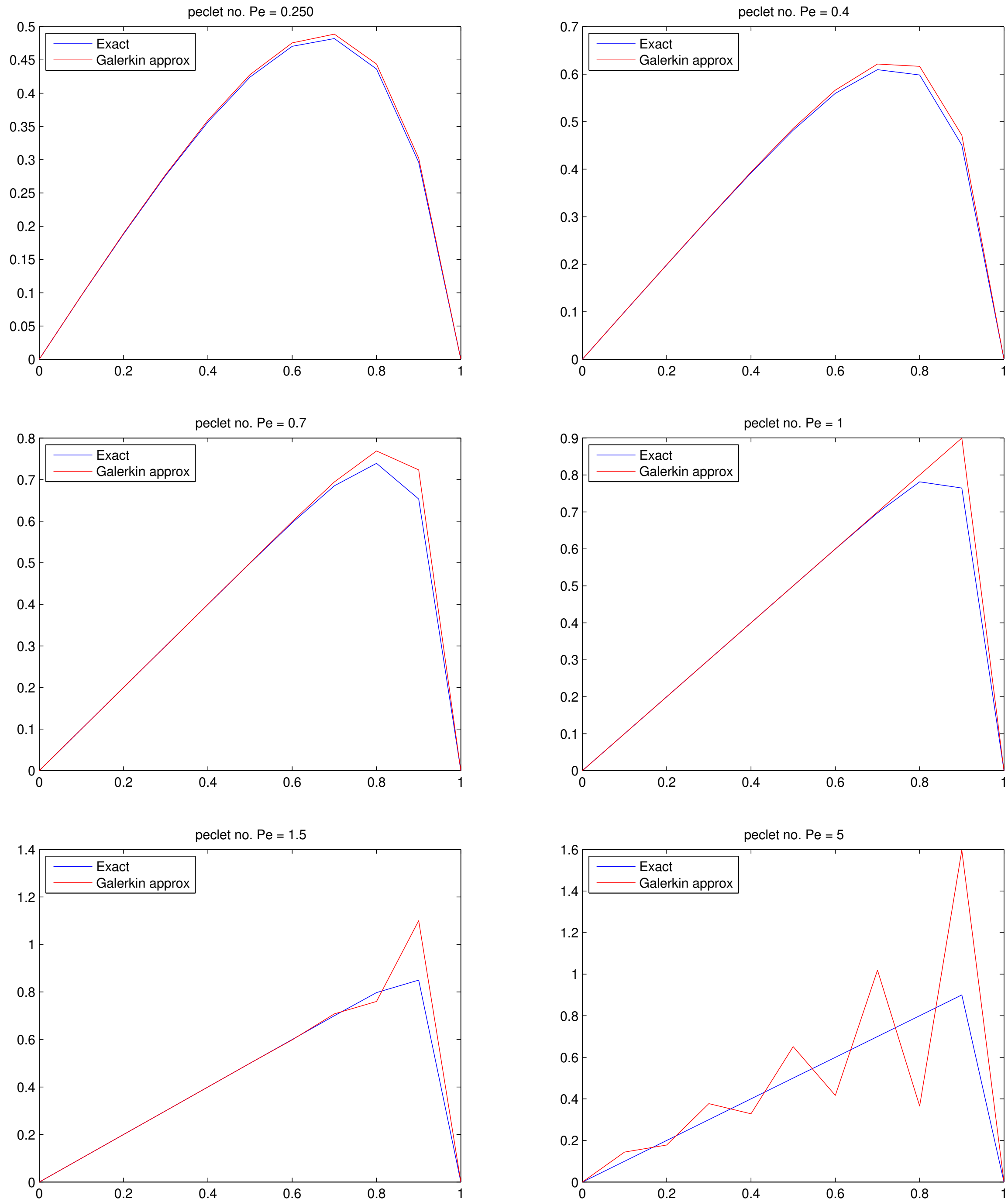


Figure 1.1: Plot of the exact solution and the GFEM solution.

1.2 The Virtual Element Method

For many practical applications, the approximation of model problem with numerical methods involving polygonal discretisation of the domain are of interest. The idea of defining finite element shape function on polygons was proposed by Wachspress in [4]. In past, many researchers have studied polytopal methods : the hybrid high-order method for linear elasticity equation problem[5],weak Galerkin FEM for Biharmonic equation [6], discontinuous Galerkin method [7]; mixed Mimetic finite difference [8], finite volume method [9] and conforming polytopal finite elements [10].

The recently introduced Virtual Element Method (VEM) is a generalisation of the Finite Element Method (FEM) that is inherently adapted to deal with arbitrary polygonal or polyhedral elements. Different from FEM is that, the finite dimensional VEM space consists of polynomial space of a specified degree and other non-polynomial functions that are locally solution of a partial differential equation. VEM is developed in such a way that the entire computation can be carried out without explicitly evaluating the basis functions. The computation of local stiffness and mass matrices are done using only the suitably defined degrees of freedom of the virtual element space. This leads to easy handling of higher order VEM and higher regularity VEM such as the more general C^α continuity for $\alpha > 1$. The VEM allows the presence of hanging nodes in the elements, use of nonconvex elements and more general adaptively refined meshes.

Since its inception VEM has been successfully applied to several problems for example linear elasticity [11], conforming and nonconforming VEM for elliptic equation [12, 13], parabolic problem [14], hyperbolic problem [15], semilinear and quasilinear problems [16–19], mixed VEM [20] for elliptic problems, acoustic vibration problem [21], Stokes problem [22], 2D magnetostatic problems [23], posteriori error estimation for the elliptic problems [24]. In a more recent paper VEM has been also applied to models of underground fluid flows [25] wherein the virtual element method becomes a more suitable approach in overcoming the mesh generation problems that is adherent in the simulation of these fluid flows. In the following section, we give a description of the virtual element method space and its associated degrees of freedom.

1.3 VEM spaces

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of partitions of Ω into polygonal elements E with h being the maximum diameter over the polygons. Despite the fact that VEM can handle arbitrary polygons, just to ensure the existence of polynomial projection operators with optimal approximation properties we consider minimal restriction on polygons. For the sake of theoretical analysis, we require $\mathcal{T}_h$ to be a quasi-uniform polygonal partitioning of Ω. To ensure the shape regularity of $\mathcal{T}_h$, assume each element $E \in \mathcal{T}_h$ satisfies the following (see [26]) :

Assumption 1.1. there exists positive constants γ and c, independent of h and E, such that

(i) E is star-shaped with respect to a disc D_γ of radius $\gamma\, h_E$, where h_E is the element diameter,

(ii) for edge $e \subset E$, the length $|e| \geq c\, h_E$,

(iii) boundary of E is made up of a finite number of edges, and $h \leq c\, h_E$.

Let $\mathbb{P}_p(E)$ denote the space of polynomials of degree $\leq p$ on E. An important component in the VEM space is the following projection operators Π_p^∇ and Π_p^0, onto the polynomial space. We define the projection operator $\Pi_p^\nabla : H^1(E) \to \mathbb{P}_p(E)$ by (see [12]),

$$\left(\nabla(u - \Pi_p^\nabla u), \nabla q_p\right)_E = 0 \quad \forall q_p \in \mathbb{P}_p(E) \text{ and } \int_{\partial E} (\Pi_p^\nabla u - u)\, ds = 0, \tag{1.3.1}$$

and Π_p^0 that is the L^2 projection onto $\mathbb{P}_p(E)$ by,

$$\left(u - \Pi_p^0 u, q_p\right)_E = 0 \quad \forall q_p \in \mathbb{P}_p(E). \tag{1.3.2}$$

Similarly, we compute the polynomial $\mathbf{\Pi}_{p-1}^0(\nabla u) \in (\mathbb{P}_{p-1}(E))^2$ by,

$$\left(\nabla u - \mathbf{\Pi}_{p-1}^0 \nabla u, \mathbf{q}_{p-1}\right)_E = 0 \quad \forall \mathbf{q}_{p-1} \in (\mathbb{P}_{p-1}(E))^2. \tag{1.3.3}$$

Consider the following space W_E^p (see [27]) for each $E \in \mathcal{T}_h$ by,

$$W_E^p = \left\{ v \in H^1(E) \cap C^0(\partial E) : v_{|e} \in \mathbb{P}_p(e)\, \forall\, \text{edge}\, e \in \partial E, \Delta v \in \mathbb{P}_p(E) \right\}.$$

Now we define the local virtual element space V_E^p as follows,

$$V_E^p = \left\{ u \in W_E^p \quad \text{s.t. } \left(u - \Pi_p^\nabla u, q\right)_E = 0 \quad \forall q \in (\mathbb{P}_p(E)/\mathbb{P}_{p-2}(E)) \right\}, \tag{1.3.4}$$

where $(\mathbb{P}_p(E)/\mathbb{P}_{p-2}(E))$ is the subspace of $\mathbb{P}_p(E)$ containing polynomials in $\mathbb{P}_p(E)$ that are L^2 orthogonal to $\mathbb{P}_{p-2}(E)$ (see [12]). We consider the following set of degrees of freedom (see Figure 1.2) on V_E^p by,

(G_1) the values of u at the $n(E)$ vertices of polygon E,

(G_2) the values of u at $p-1$ internal Gauss-Lobatto quadrature nodes of every edge $e \in \partial E$,

(G_3) the moments up to order $p-2$ of u in E, i.e.,

$$\int_E u\, q_{p-2}\, dx \quad \forall q_{p-2} \in \mathbb{P}_{p-2}(E).$$

We note that the degrees of freedom mentioned above determine u uniquely on the polygon E, (see [27]). Therefore the dimension of the local space is given by the formula,

$$\dim V_E^p := p\, N_E + \frac{p(p-1)}{2},$$

where N_E is the number of vertices in polygon E.

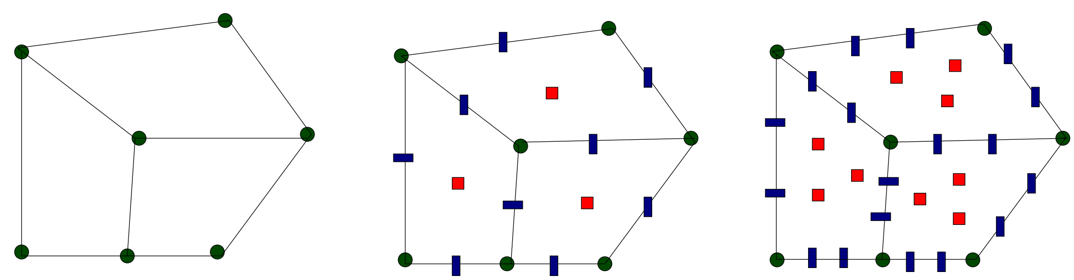


Figure 1.2: Degrees of freedom for $k = 1, 2, 3$ (from left to right). We denote G_1 by green circle, G_2 by blue rectangle and the moments G_3 by red square.

Now let us define the global virtual element space V_h^p by,

$$V_h^p = \{u \in H_0^1(\Omega) \text{ s.t. } u_{|E} \in V_E^p \; \forall E \in \mathcal{T}_h\}. \tag{1.3.5}$$

Note that the polynomial space $\mathbb{P}_p(E)$ is a subspace of the local VEM space V_E^p. The core principles of VEM are elaborately discussed in [26]. We remark that the projection operators Π_p^∇ and Π_p^0 defined on the Sobolev space $H^1(E)$ are computable on the VEM space V_h^p, and not on the general spaces $H^1(E)$ (refer [27]).

1.3.1 Role of the operators Π_p^∇ and Π_p^0

The VEM space contains the polynomial space and a set of non-polynomial functions that are solution of certain partial differential equation. The explicit definition of the non-polynomial functions are not known and are never required for the computation purposes. We demonstrate the role of the operators Π_p^{∇} and Π_p^0 in the discrete formulation in guaranteeing the *VEM computablity* of the inner products in the scheme. By *VEM computablity* we mean, the term is evaluated just using the degrees of freedom and the polynomial components of the functions.

For illustration, we discuss two variety of the approximation of the gradient inner product or the stiffness term $a(u,v) := (\nabla u,\, \nabla v)_{\Omega}$.

Type I : Using only Π_p^{∇}

For a function $u_h \in V_E^p$, we split u_h into its polynomial and non-polynomial counterpart, using the polynomial projection operators Π_p^{∇} as follows :

$$u_h \;=\; \Pi_p^{\nabla} u_h + u_h - \Pi_p^{\nabla} u_h.$$

Therefore, using the property (1.3.1), we have

$$\begin{aligned} a(u_h,\, v_h)_E \;=\;& (\nabla \Pi_p^{\nabla} u_h,\, \nabla \Pi_p^{\nabla} v_h) + (\nabla u_h - \nabla \Pi_p^{\nabla} u_h,\, \nabla \Pi_p^{\nabla} v_h) \\ & + (\nabla \Pi_p^{\nabla} u_h,\, \nabla v_h - \nabla \Pi_p^{\nabla} v_h) + (\,\nabla u_h - \nabla \Pi_p^{\nabla} u_h,\, \nabla v_h - \nabla \Pi_p^{\nabla} v_h) \\ =\;& (\nabla \Pi_p^{\nabla} u_h,\, \nabla \Pi_p^{\nabla} v_h) + (\,\nabla u_h - \nabla \Pi_p^{\nabla} u_h,\, \nabla v_h - \nabla \Pi_p^{\nabla} v_h). \end{aligned}$$

For each $E \in \mathcal{T}_h$, the discrete bilinear form $a_h^E(\cdot,\cdot) : V_E^p \times V_E^p \to \mathbb{R}$ is defined as follows : $\forall\, u_h,\, v_h \in V_p^E$,

$$a_h^E(u_h, v_h) := a^E\Big(\Pi_p^{\nabla} u_h, \Pi_p^{\nabla} v_h\Big) + S_a^E\Big((I - \Pi_p^{\nabla})u_h, (I - \Pi_p^{\nabla})v_h\Big), \tag{1.3.6}$$

where $S_a^E(\cdot,\cdot)$ is a symmetric positive definite bilinear form which ensures stability of discrete bilinear form $a_h^E(\cdot,\cdot)$, that is, there exists constants $0 < \mu_* \le \mu^*$, independent of h_E, such that,

$$\mu_*\, a^E(u_h, u_h)_E \le S_a^E(u_h, u_h) \le \mu^*\, a^E(u_h, u_h) \qquad \forall u_h \in \ker(\Pi_p^{\nabla}).$$

Finally, the global bilinear form $a_h(\cdot,\cdot) : V_h^p \times V_h^p \to \mathbb{R}$ that approximates the stiffness term $a(\cdot,\cdot)$ is defined such that

$$a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} a_h^E(u_h, v_h)\; \forall\, u_h, v_h \in V_h^p. \tag{1.3.7}$$

Type II : Using $\mathbf{\Pi}_{p-1}^0$ and Π_p^{∇}

We split $\nabla u_h := \mathbf{\Pi}^0_{p-1}\nabla u_h + \nabla u_h - \mathbf{\Pi}^0_{p-1}\nabla u_h$. Then we have using (1.3.1) and (1.3.2),

$$\begin{aligned} a(u_h,\, v_h)_E &= (\mathbf{\Pi}^0_{p-1}\nabla u_h,\, \mathbf{\Pi}^0_{p-1}\nabla v_h) + (\nabla u_h - \mathbf{\Pi}^0_{p-1}\nabla u_h,\, \mathbf{\Pi}^0_{p-1}\nabla v_h) \\ &\quad (\mathbf{\Pi}^0_{p-1}\nabla u_h,\, \nabla v_h - \mathbf{\Pi}^0_{p-1}\nabla v_h) + (\nabla u_h - \mathbf{\Pi}^0_{p-1}\nabla u_h,\, \nabla v_h - \mathbf{\Pi}^0_{p-1}\nabla v_h) \\ &= (\mathbf{\Pi}^0_{p-1}\nabla u_h,\, \mathbf{\Pi}^0_{p-1}\nabla v_h) + (\nabla u_h - \mathbf{\Pi}^0_{p-1}\nabla u_h,\, \nabla v_h - \mathbf{\Pi}^0_{p-1}\nabla v_h) \\ &= (\mathbf{\Pi}^0_{p-1}\nabla u_h,\, \mathbf{\Pi}^0_{p-1}\nabla v_h) + (\nabla u_h - \nabla\Pi^\nabla_p u_h,\, \nabla v_h - \nabla\Pi^\nabla_p v_h). \end{aligned}$$

Similarly, for each $E \in \mathcal{T}_h$, the discrete bilinear form $\widetilde{a}^E_h(\cdot,\cdot) : V^p_E \times V^p_E \to \mathbb{R}$ is defined as follows : $\forall\, u_h,\, v_h \in V^E_p$,

$$\widetilde{a}^E_h(u_h, v_h) := \Big(\mathbf{\Pi}^0_{p-1}\nabla u_h,\, \mathbf{\Pi}^0_{p-1}\nabla v_h\Big) + S^E_a\Big((I - \Pi^\nabla_p)u_h, (I - \Pi^\nabla_p)v_h\Big), \tag{1.3.8}$$

where $S^E_a(\cdot,\cdot)$ is a symmetric positive definite bilinear form which ensures stability of discrete bilinear form $a^E_h(\cdot,\cdot)$. Finally, the global bilinear form $\widetilde{a}_h(\cdot,\cdot) : V^p_h \times V^p_h \to \mathbb{R}$ that approximates the stiffness term $a(\cdot,\cdot)$ is defined such that

$$\widetilde{a}_h(u_h, v_h) := \sum_{K\in\mathcal{T}_h} \widetilde{a}^E_h(u_h, v_h)\ \forall\, u_h, v_h \in V^p_h. \tag{1.3.9}$$

Among $a_h(\cdot,\cdot)$ and $\widetilde{a}_h(\cdot,\cdot)$ the choice of the discrete bilinear form to be considered for approximating $a(\cdot,\cdot)$ is problem dependent. Through deep analysis, we can determine the suitable discrete form that does not affect the rate of convergence (refer [28]).

A detailed procedure for the computation of the operators $\Pi^\nabla_p u_h$ and $\Pi^0_p u_h$ is given in [29] and the estimation of $\mathbf{\Pi}^0_{p-1}\nabla u_h$ is discussed in [30].

1.4 Motivation

The convection-diffusion-reaction equation governing many practical situations has a complex domain under consideration, for example, a fractured networks such as in underground water channels or a domain with internal substructures as in cellular biology. For complicated domains, use of polygonal elements for discretisation is more desirable. Both the Polygonal Finite Element Method (PFEM) and the virtual element method can accommodate elements with arbitrary shapes and sizes, however, one distinct feature of the VEM when compared to the PFEM is that the later requires an explicit form of the basis functions to compute the bilinear and the linear forms. The basis functions over arbitrary polytopes are rational polynomials, and thus computation requires higher order numerical quadrature rules. Whilst in case of the VEM, no such explicit form of the basis functions is required and moreover, higher order elements even in higher dimensions can easily be constructed. Therefore virtual element method is highly suitable for numerically solving problems entailing complex domains. Most often in transport problems, the convection part is the most dominating. Under these circumstances, stabilization of the virtual element method is imperative. For linear convection-diffusion equation, the streamline upwind Petrov-Galerkin stabilization of VEM was analysed and a priori estimate with optimal order of convergence was derived in [31]. Our main objective is to conduct theoretical and numerical study of residual based stabilization of the virtual element method for approximating the nonlinear convection-diffusion-reaction problems.

1.5 Notations and Preliminaries

Let us consider a bounded domain Ω subset of $\mathbb{R}^2$. We state the necessary notations and mathematical tools used in the thesis.

Definition 1.1. For $p \in \mathbb{N},$ a measurable function f defined on Ω is called $p-integrable$ if it satisfies,

$$\|f\|_{L^p(\Omega)} := \Big(\int\limits_{\Omega} |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty.$$

Definition 1.2. The $L^p(\Omega)$ space is the collection of all $p - integrable$ functions defined on Ω. That is,

$$L^p(\Omega) := \Big\{ f : \Omega \to \mathbb{R} \ \Big| \ \|f\|_{L^p(\Omega)} < \infty \Big\}.$$

For $p = \infty$, the space $L^\infty(\Omega)$ consists of all essentially bounded functions on Ω with the norm,

$$\|f\|_\infty = \text{ess}\sup_{x\in\Omega} |f(x)| := \inf_{\omega\subset\Omega, |\omega|=0} \sup_{\Omega\setminus\omega} f(x).$$

Definition 1.3. We define the following inner product on the $L^2(\Omega)$:

$$(\cdot,\cdot)_\Omega := \|\cdot\|^2_{L^2(\Omega)}.$$

Let us denote by $\mathbf{m} = (m_1, m_2, \ldots, m_d)$, a d-tuple multi-index of non-negative integers m_i with its order m defined by $m = \sum_{i=1}^{d} m_i$. Then the m^{th} order partial derivative is defined as

$$D^{\mathbf{m}} = \frac{\partial^{\mathbf{m}}}{\partial x_1^{m_1} \cdots \partial x_d^{m_d}}.$$

Definition 1.4. (Sobolev space of order (s, p) over Ω)

For a non-negative integer s and $1 \leq p \leq \infty$, the space is $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) : D^{\mathbf{m}}u \in L^p(\Omega), m \leq s\},$$

equipped with the norm

$$\|u\|_{s,p} = \Big(\sum_{m \leq S} \|D^{\mathbf{m}}u\|^p_{L^p(\Omega)}\Big)^{1/p} \quad \forall\, 1 \leq p < \infty, \quad \|u\|_{s,\infty} = \sup_{m \leq s} \|D^{\mathbf{m}}u\|_{L^\infty(\Omega)} \quad p = \infty.$$

and the semi-norm is defined as

$$|u|_{s,p} = \Big(\sum_{m = s} \|D^{\mathbf{m}}u\|^p_{L^p(\Omega)}\Big)^{1/p} \quad \forall\, 1 \leq p < \infty, \quad |u|_{s,\infty} = \sup_{m = s} \|D^{\mathbf{m}}u\|_{L^\infty(\Omega)} \quad p = \infty.$$

For $p = 2$, the Sobolev space $W^{s,2}(\Omega)$ is an inner product space and is usually denoted as $H^s(\Omega)$. Let us denote $H^0(\Omega) := L^2(\Omega)$. To impose clarity on the domain, we use the following notations for the norm and semi-norm defined on $H^s(\Omega)$. That is,

$$\|\cdot\|_{s,\Omega} := \|\cdot\|_{s,2} \quad \text{and} \quad |\cdot|_{s,\Omega} := |\cdot|_{s,2}.$$

Definition 1.5. The $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ and the dual of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$.

Proposition 1.1. *(Young's inequality)*

$$ab \leq \frac{\epsilon\, a^2}{2} + \frac{b^2}{2\epsilon} \qquad \forall\, a, b \in \mathbb{R}^+ \cup \{0\},\ \epsilon \in \mathbb{R}^+.$$

Proposition 1.2. *(Hölder's inequality for sums)*
Let $\dfrac{1}{p} + \dfrac{1}{q} = 1$ *with* $p,\ q > 1$. *For* $a_1, ..., a_n, b_1, ..., b_n$ *be real numbers. Then*

$$\sum_{i=1}^n |a_i b_i| \leq \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |b_i|^q\right)^{\frac{1}{q}}.$$

Proposition 1.3. *(Hölder's inequality for integrals)*

Let $\dfrac{1}{p}+\dfrac{1}{q}=1$ *with* $p,\ q>1$. *For* $f\in L^p(\Omega)$ *and* $g\in L^q(\Omega)$. *Then* $fg\in L^1(\Omega)$ *and*

$$\|fg\|_{L^1(\Omega)}\ \leq\ \|f\|_{L^p(\Omega)}\,\|g\|_{L^q(\Omega)}.$$

When p=q=2, this inequality is known as Cauchy-Schwarz inequality.

Proposition 1.4. *(Generalised Hölder's inequality)*
Let $r,p_1,p_2,...,p_n\in\mathbb{R}^+\cup\{\infty\}$ *with* $\dfrac{1}{\infty}=0,$ *satisfy* $\sum_{i=1}^{n}\dfrac{1}{p_i}=r$. *Then for any collection of* $f_i's,\ i=1,...,n$ *with* $f_i\in L^{p_i}(\Omega)$*, we have the relation :*

$$\|\prod_{i=1}^{n}f_i\|_{L^r(\Omega)}\ \leq\ \prod_{i=1}^{n}\|f_i\|_{L^{p_i}.(\Omega)}$$

1.6 Outline of the thesis

In chapter 2, we study the SUPG stabilization of VEM for a semilinear convection-diffusion-reaction equation. We propose a computable VEM scheme, discuss the well-posedness and optimal order convergence estimate concerning the energy norm. Chapter 3 deals with the discussion of the shock-capturing stabilization of VEM. First, we propose a computable discrete scheme for the linear convection-diffusion-reaction equation and show the existence of a numerical solution. Then we investigate the efficiency of the shock-capturing method through numerical simulations. Subsequently, we devise a shock-capturing stabilized VEM formulation for the semilinear convection-diffusion-reaction equation. We prove the existence of a discrete solution. An optimal order convergence estimate with respect to the energy norm is derived, and numerical simulations are presented to illustrate the efficiency of the added stabilizer. Chapter 4 treats the SUPG stabilization of VEM for the quasilinear convection-diffusion-reaction equation. We discuss the well-posedness of the discrete scheme by approximation of branches of nonsingular solution. We derive an optimal convergence estimate with respect to the energy norm. In simulations, we discuss using a two-grid method to reduce the CPU time taken to solve the discrete system. In Chapter 5, We study the VEM for the nonlocal coupled reaction-diffusion equation. We discuss the well-posedness of the fully discrete scheme and derive an optimal order convergence estimate with respect to the L^2 norm and H^1 seminorm. Also, to restore the sparsity structure of Newton's Jacobian, we suggest a remedy. Numerical experiments validating the theoretical estimates are presented. In the final chapter, we discuss the future scope for the topics discussed in chapters 2-5.

Chapter 2

Virtual element method for the semilinear convection-diffusion-reaction equation on polygonal meshes

Nonlinear convection-diffusion-reaction equation arises in all branches of science and engineering. Some important practical models include combustion of subsurface reactive transport processes, movement of fluids in porous solids [32], drift-diffusion equations of semiconductor device modelling, heat transfer problems [33] and heat-induced moisture transport in porous media [34]. The convection-diffusion equation is a prototype of the nonlinear Navier-Stokes equation of fluid flows. An explicit analytical solution is not at one's disposal for the model partial differential equations of these types with sophisticated boundary conditions. Hence, researchers are interested in obtaining an efficient approximate solution for the convection-diffusion equations. We know that wild oscillations appear in the numerical solution obtained from the standard discretisation techniques for the singularly perturbed problems. The spurious oscillations occur in the neighbourhood of layers of the solution. It is still challenging to devise and analyse a discrete formulation that solves with optimal accuracy when the problem is either convection-dominated or reaction-dominated. To overcome these situations several stabilization techniques have been proposed in the literature, for example, Streamline upwind Petrov-Galerkin (SUPG) [35, 36], local projection stabilization [37, 38], edge stabilization [39].

In many industrial and practical situations, the concerned domains are of complex geometry. For such instances, meshing using polygonal elements is highly advantageous. Few numerical analysis involving arbitrary polygonal or polyhedral meshes can be found in [4],[40],[41] and [42]. We can note that for these existing techniques, the method does not allow non-convex elements and degenerating elements (i.e. hanging node-like structures) in the domain discretisation. One requires performing numerical integration using the quadrature formulas for evaluating the associated bilinear forms. The requirement of knowing explicit canonical basis functions over polygonal elements makes the implementation of higher-order methods more inconvenient.

The virtual element method is naturally adapted to general polygonal or polyhedral meshes. A significant feature of VEM is that it can handle meshes without an explicit definition of shape functions. The VEM space with its associated degrees of freedom is defined such that to obtain polynomial accurate and stable numerical solutions; we only require the information about polynomial subspaces of local virtual element space. Suitable polynomial projection operators are introduced into the discrete formulation to ensure that computations are carried out only using the VEM degrees of freedom. In [29], it is noted for diffusion or diffusion-reaction problems, VEM analysis does not involve numerical quadrature formulas. Moreover, VEM allows the use of arbitrary polygonal meshes with hanging nodes; that is, the angle between two edges can be $180°$. This characteristic feature of VEM makes it more suitable for approximating problems involving the generation of conforming adaptive meshes, for example, in fluid dynamics such as underground flow problems [25]. In most flow/transport problems, the convection phenomenon is often more dominating than the diffusion counterpart. Therefore, stabilization of the virtual element method is of primary interest. In literature, for linear convection-diffusion equations, SUPG stabilization of conforming and non-conforming virtual element method [31], [43]; local projection stabilization of VEM [44] and VEM stabilization using link-cutting conditions [45] are discussed. This chapter proposes and analyses virtual element discretisation of the nonlinear convection-diffusion-reaction equation with SUPG stabilization.

2.1 Governing equation and weak formulation

Let us consider the following nonlinear convection-diffusion-reaction problem

$$\begin{aligned} &\sigma\, u - \nabla \cdot (K\, \nabla u) + \mathbf{b} \cdot \nabla u + g(u) = f \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.1.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^2$ with the following assumptions:
(A.1a) constant $\sigma > 0$, $K \in L^\infty(\Omega)$ satisfying $K(x) \geq K_0 > 0$ a.e in Ω,
(A.1b) $\mathbf{b} \in (W^{1,\infty}(\Omega))^2$ with $(\nabla \cdot \mathbf{b})(x) = 0$ a.e in Ω,
(A.1c) $g \in C^1(\mathbb{R})$ with $g(0) = 0$ and $g'(x) \geq g_0 \geq 0$ for $x \geq 0$ and $f \in L^2(\Omega)$.

For the purpose of practical applications such as concentration of pollutants, we consider the solution u to be non-negative and bounded above i.e., $u_0 \leq u \leq u_1$ with $u_0 \geq 0$. From (A.1c) we note that g' is bounded on compact intervals of u. This implies g is Lipschitz continuous with constant L. The standard variational formulation of the continuous problem (2.1.1) is given by : Find $u \in H_0^1(\Omega)$ such that

$$A(u,v) = (f,v)_\Omega \quad \forall v \in H_0^1(\Omega), \tag{2.1.2}$$

where, $A(u,v) = (\sigma u, v)_\Omega + (K\,\nabla u, \nabla v)_\Omega + (\mathbf{b}\cdot\nabla u, v)_\Omega + (g(u),v)_\Omega$. We formulate (2.1.2) as an operator equation $\mathcal{A}u = f$, where the operator $\mathcal{A} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ satisfies $(\mathcal{A}u, v)_\Omega = A(u,v)\ \forall u, v \in H_0^1(\Omega)$. Using the assumptions (A.1a-c), we obtain $\forall u, v \in H_0^1(\Omega)$,

$$\begin{aligned} (\mathcal{A}(u) - \mathcal{A}(v), (u-v))_\Omega &\geq \min\{\sigma + g_0\,;\, K_0\}\, \|u-v\|_{1,\Omega} \\ \|\mathcal{A}u - \mathcal{A}v\|_{H^{-1}(\Omega)} &\leq \max\{(\sigma + L)\,;\, (\|\mathbf{b}\|_{\infty,\Omega} + \|K\|_{\infty,\Omega})\}\, \|u-v\|_{1,\Omega}. \end{aligned}$$

Thus the operator $\mathcal{A}$ is strongly monotone and Lipschitz continuous which implies that the operator equation has a unique solution [46].

2.2 VEM-SUPG stabilization

When the problem is singularly perturbed, the standard numerical methods approximating (2.1.2) generate solutions affected with spurious oscillations. To overcome this situation, a stabilization of VEM is required. In this section we formulate the Streamline upwind Petrov-Galerkin (SUPG) stabilization for the VEM discretization. In VEM, as mentioned earlier, the functions in V_h^p are not known explicitly in the interior of elements $E \in \mathcal{T}_h$. Hence to guarantee the computablity of the virtual element formulation, we use the projection operators Π_p^0, Π_p^∇ and $\mathbf{\Pi}_{p-1}^0$ in the approximation of (2.1.2). The introduction of projection operators alters the skew-symmetric property of the term $(\mathbf{b}\cdot\nabla u, v)_\Omega$. Therefore, using the assumption (A.1b), we consider an equivalant form as follows :

$$(\mathbf{b}\cdot\nabla u, v)_\Omega := \frac{1}{2}(\mathbf{b}\cdot\nabla u,\, v)_\Omega - \frac{1}{2}(\mathbf{b}\cdot\nabla v,\, u)_\Omega. \tag{2.2.1}$$

The modification (2.2.1) helps to preserve the skew-symmetric property in VEM and moreover, this helps us to prove the well-posedness of the virtual element scheme irrespective of mesh diameter h (unlike in [31] where one requires sufficiently small h).

The terms in the VEM are defined as follows, one by one.

$$
\begin{aligned}
a(u_h, v_h) &:= \left(K\, \mathbf{\Pi}^0_{p-1}\nabla u_h,\, \mathbf{\Pi}^0_{p-1}\nabla v_h\right)_\Omega + \sum_{E\in\mathcal{T}_h} \tau_E \left(\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla u_h,\, \mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\right)_E \\
&\quad + \sum_{E\in\mathcal{T}_h} \left(K_E + \tau_E \mathbf{b}_E^2\right) S_1^E\left(\left(I - \Pi_p^\nabla\right) u_h, \left(I - \Pi_p^\nabla\right) v_h\right), \qquad (2.2.2)\\
b(u_h, v_h) &:= \left(\sigma\, \Pi_p^0 u_h,\, \Pi_p^0 v_h\right)_\Omega + \sum_{E\in\mathcal{T}_h} \sigma\, S_2^E\left((I-\Pi_p^0)u_h, (I-\Pi_p^0)v_h\right), \qquad (2.2.3)\\
c(u_h, v_h) &:= \frac{1}{2}\left[\left(\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla u_h,\, \Pi_p^0 v_h\right)_\Omega - \left(\Pi_p^0 u_h,\, \mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\right)_\Omega\right] \\
&\quad + \sum_{E\in\mathcal{T}_h} \tau_E \left(\sigma\, \Pi_p^0 u_h - \nabla\cdot K\mathbf{\Pi}^0_{p-1}\nabla u_h,\; \mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\right)_E, \qquad (2.2.4)\\
d(u_h, v_h) &:= \left(\hat{g}(\Pi_p^0 u_h),\, \Pi_p^0 v_h\right)_\Omega + \sum_{E\in\mathcal{T}_h} g_0\, S_2^E\left((I-\Pi_p^0)u_h, (I-\Pi_p^0)v_h\right) \\
&\quad + \sum_{E\in\mathcal{T}_h} \tau_E \left(\hat{g}(\Pi_p^0 u_h),\, \mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\right)_E, \qquad (2.2.5)\\
F_{vsg}(v_h) &:= \left(f, \Pi_p^0 v_h\right)_\Omega + \sum_{E\in\mathcal{T}_h} \tau_E \left(f, \mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\right)_E, \qquad (2.2.6)
\end{aligned}
$$

where $K_E := \sup_{x\in E} K(x)$, $\mathbf{b}_E := \sup_{x\in E} \|\mathbf{b}(x)\|_{0,\mathbb{R}^2}$ and let $K_E^\vee := \inf_{x\in E} K(x)$. Then we define,

$$
A_{vsg}(u_h, v_h) := a(u_h, v_h) + b(u_h, v_h) + c(u_h, v_h) + d(u_h, v_h). \qquad (2.2.7)
$$

We state the VEM-SUPG discrete formulation as : Find $u_h \in V_h^p$ such that

$$
A_{vsg}(u_h, v_h) = F_{vsg}(v_h) \quad \forall v_h \in V_h^p. \qquad (2.2.8)
$$

Whenever $g'(\cdot)$ is not bounded above in $\mathbb{R}^+$ we use $\hat{g}(\cdot)$ in place of $g(\cdot)$ defined as follows,

$$
\hat{g}(u) = \begin{cases} g(u_0) + g'(u_0)(u - u_0) & \text{for} \quad u \le u_0. \\ g(u) & \text{for} \quad u_0 \le u \le u_1. \\ g(u_1) + g'(u_1)(u - u_1) & \text{for} \quad u \ge u_1. \end{cases} \qquad (2.2.9)
$$

Note that S_1^E and S_2^E denote the symmetric bilinear form defined on $V_E^p \times V_E^p$. Let there exists non-zero positive constants λ_*, λ^*, μ_* and μ^*, with $\lambda_* \le \lambda^*$ and $\mu_* \le \mu^*$, independent of h_E, such that,

$$\lambda_*(\nabla u_h, \nabla u_h)_E \leq S_1^E(u_h, u_h) \leq \lambda^*(\nabla u_h, \nabla u_h)_E \quad \forall u_h \in \ker(\Pi_p^\nabla), \quad (2.2.10)$$
$$\mu_*(u_h, u_h)_E \leq S_2^E(u_h, u_h) \leq \mu^*(u_h, u_h)_E \quad \forall u_h \in \ker(\Pi_p^0). \quad (2.2.11)$$

We consider the following choice for computational purposes :

$$S_1^E(u_h, v_h) = \sum_{i=1}^{\text{ndof}} \text{dof}_i(u_h)\, \text{dof}_i(v_h) \quad \text{and} \quad S_2^E(u_h, v_h) = h_E^2 \sum_{i=1}^{\text{ndof}} \text{dof}_i(u_h)\, \text{dof}_i(v_h),$$

where $\text{dof}_i(u_h)$ denotes the ith degrees of freedom of u_h with ndof denoting the total degrees of freedom of E. Since the degrees of freedom scales like 1, suitable scaling coefficients are used for the stabilization terms S_1^E and S_2^E respectively (see [13]). The stabilization term that appears in (2.2.5) is useful in deriving the coercivity estimate.

2.3 Error estimates

In this section we derive the error estimates for the proposed VEM-SUPG discretization. Our analysis will follow in similar lines to the finite element error analysis performed in [47]. Considering the assumptions of g mentioned in section 2.1 and the definition of $\hat{g}$ (see (2.2.9)) we note that $\hat{g}'(u)$ is bounded on the compact intervals of u. This implies that $\hat{g}(\cdot)$ is Lipschitz continuous with constant L_g.

2.3.1 Preliminary results

We prove the coercivity estimate followed by existence and uniqueness of the discrete solution. The following polynomial inverse inequality is given in [48]. Let $q \in \mathbb{P}_r(E)$ with $r \in \mathbb{N} \cup \{0\}$. Then

$$\|q\|_{0,E} \leq c\frac{(r+1)^2}{h_E}\|q\|_{-1,E}, \quad (2.3.1)$$

where c is a positive constant independent of h_E, r, and $\|q\|_{-1,E} := \|q\|_{(H_0^1(E))^*}$. For $u_h \in V_h^p$, we know that $\Delta u_h \in \mathbb{P}_p(E)\ \forall E \in \mathcal{T}_h$. Taking $q = \Delta u_h$ in (2.3.1) and using the estimate $\|\Delta u_h\|_{-1,E} \leq |u_h|_{1,E}$ given in [49], we get,

$$\|\Delta u_h\|_{0,E} \leq \mu_{\text{inv}}\, p^2\, h_E^{-1}\, |u_h|_{1,E}. \quad (2.3.2)$$

where, $\mu_{\text{inv}} > 0$ is independent of u_h, h_E and p. We introduce the following norm

$$|||u|||^2 := \sum_{E\in\mathcal{T}_h} \Big(\|\sqrt{K}\,\nabla u\|_{0,E}^2 + (\sigma + g_0)\,\|u\|_{0,E}^2 + \tau_E\,\|\mathbf{b}.\nabla u\|_{0,E}^2 \Big).$$

Lemma 2.1 (coercivity)**.** *Let* $0 \leq \tau_E \leq \dfrac{1}{4}min\left\{ \dfrac{h_E^2}{p^4\mu_{inv}^2 K_E}, \dfrac{1}{\sigma}, \dfrac{\sigma+g_0}{L_g^2} \right\}$ *be satisfied. Then,*

$$A_{vsg}(v_h, v_h) \geq \theta\,|||v_h|||^2 \qquad \forall v_h \in V_h^p, \tag{2.3.3}$$

where, $\theta = \min\left\{\frac{1}{4},\, \lambda_*,\, \mu_*\right\}$.

Proof. We bound the terms of $A_{vsg}(\cdot,\cdot)$ one by one. Considering the first term,

$$\begin{aligned} a(v_h, v_h) = \sum_{E\in\mathcal{T}_h} \Big(&\|\sqrt{K}\,\mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E}^2 + \tau_E \|\mathbf{b}\cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E}^2 \\ &+ \left(K_E + \tau_E \mathbf{b}_E^2\right)\, S_1^E\left((I-\Pi_p^\nabla)v_h, (I-\Pi_p^\nabla)v_h\right) \Big). \end{aligned}$$

Using (2.2.10) and inequality $\|(I-\mathbf{\Pi}_{p-1}^0)\nabla u_h\|_{0,E} \leq \|\nabla(I-\Pi_p^\nabla)u_h\|_{0,E}$ ([12]), we get,

$$\begin{aligned} a(v_h, v_h) &\geq \sum_{E\in\mathcal{T}_h} \Big(\|\sqrt{K}\mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E}^2 + \tau_E \|\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E}^2 \\ &\quad + \left(K_E + \tau_E \mathbf{b}_E^2\right)\lambda_* \|\nabla(I-\Pi_p^\nabla)v_h\|_{0,E}^2 \Big) \\ &\geq \sum_{E\in\mathcal{T}_h} \Big(\|\sqrt{K}\,\mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E}^2 + \lambda_* \|\sqrt{K}(I-\mathbf{\Pi}_{p-1}^0)\nabla v_h\|_{0,E}^2 \\ &\quad + \tau_E \|\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E}^2 + \lambda_* \tau_E \|\mathbf{b}\cdot(I-\mathbf{\Pi}_{p-1}^0)\nabla v_h\|_{0,E}^2 \Big). \end{aligned} \tag{2.3.4}$$

Using the inequality (2.2.11), we obtain,

$$\begin{aligned} b(v_h, v_h) &= \sum_{E\in\mathcal{T}_h} \Big(\|\sqrt{\sigma}\,\Pi_p^0 v_h\|_{0,E}^2 + \sigma\, S_2^E\left((I-\Pi_p^0)v_h, (I-\Pi_p^0)v_h\right) \Big) \\ &\geq \sum_{E\in\mathcal{T}_h} \sigma \|\Pi_p^0 v_h\|_{0,E}^2 + \mu_* \sum_{E\in\mathcal{T}_h} \sigma \|(I-\Pi_p^0)v_h\|_{0,E}^2. \end{aligned} \tag{2.3.5}$$

Consider $|c(v_h, v_h)| = \Big| 0 + \sum_{E\in\mathcal{T}_h} \tau_E \Big(\sigma\,\Pi_p^0 v_h - \nabla\cdot K\mathbf{\Pi}_{p-1}^0 \nabla v_h, \mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0 \nabla v_h \Big)_E \Big|$.
Applying triangle inequality and Cauchy-Schwarz inequality, we get,

$$\begin{aligned} c(v_h, v_h) \leq \sum_{E\in\mathcal{T}_h} \Big(&\tau_E\,\sigma\,\|\Pi_p^0 v_h\|_{0,E} \|\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E} \\ &+ \tau_E \|\nabla\cdot K\mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E} \|\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E} \Big). \end{aligned}$$

Using $\tau_E \leq \dfrac{1}{4\sigma}$, we get, $\sigma \, \|\Pi_p^0 v_h\|_{0,E} \leq \dfrac{\sqrt{\sigma}}{2\sqrt{\tau_E}} \|\Pi_p^0 v_h\|_{0,E}$. Similarly, using the inverse inequality (2.3.2) and then using $\tau_E \leq \dfrac{1}{4} \dfrac{h_E^2}{p^4 \mu_{\text{inv}}^2 K_E}$, we get,

$$\|\nabla \cdot K \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E} \leq \frac{1}{2\sqrt{\tau_E}} \|\sqrt{K} \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E}. \tag{2.3.6}$$

Thus, we obtain,

$$\begin{aligned}
|c(v_h, v_h)| \leq & \sum_{E \in \mathcal{T}_h} \Big(\frac{1}{2} \sqrt{\sigma} \|\Pi_p^0 v_h\|_{0,E} \sqrt{\tau_E} \|\mathbf{b} \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E} \\
& + \frac{1}{2} \|\sqrt{K} \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E} \sqrt{\tau_E} \|\mathbf{b} \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E} \Big) \\
& = \sum_{E \in \mathcal{T}_h} \Big\{ \Big(\sqrt{\sigma} \|\Pi_p^0 v_h\|_{0,E} \Big) \left(\frac{\sqrt{\tau_E}}{2} \|\mathbf{b} \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E} \right) \\
& + \Big(\frac{1}{\sqrt{2}} \|\sqrt{K} \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E} \Big) \left(\frac{\sqrt{\tau_E}}{\sqrt{2}} \|\mathbf{b} \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E} \right) \Big\}.
\end{aligned}$$

Using Young's inequality for products, $mn \leq \dfrac{m^2}{2} + \dfrac{n^2}{2}$, we get,

$$|c(v_h, v_h)| \leq \sum_{E \in \mathcal{T}_h} \left(\frac{\sigma}{2} \|\Pi_p^0 v_h\|_{0,E}^2 + \frac{\tau_E}{2} \|\mathbf{b} \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E}^2 + \frac{1}{4} \|\sqrt{K} \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E}^2 \right).$$

Thus we get,

$$\begin{aligned}
c(v_h, v_h) \geq & - \sum_{E \in \mathcal{T}_h} \Big(\frac{\sigma}{2} \|\Pi_p^0 v_h\|_{0,E}^2 + \frac{\tau_E}{2} \|\mathbf{b} \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E}^2 \\
& + \frac{1}{4} \|\sqrt{K} \mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E}^2 \Big). \qquad (2.3.7) \\
d(v_h, v_h) = & \left(\hat{g}(\Pi_p^0 v_h), \Pi_p^0 v_h \right) + \sum_{E \in \mathcal{T}_h} g_0 \, S_2^E \left((I - \Pi_p^0) v_h, (I - \Pi_p^0) v_h \right) \\
& + \sum_{E \in \mathcal{T}_h} \tau_E \left(\hat{g}(\Pi_p^0 v_h), \mathbf{b} \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \right)_E. \qquad (2.3.8)
\end{aligned}$$

Considering the first term of (2.3.8), we note

$$\left(\hat{g}(\Pi_p^0 v_h), \Pi_p^0 v_h \right) \geq \frac{g_0}{2} \sum_{E \in \mathcal{T}_h} \|\Pi_p^0 v_h\|_{0,E}^2, \tag{2.3.9}$$

which can be derived from $\hat{g}(x)\, x = (\hat{g}(x) - \hat{g}(0))\; x \geq \int_0^x s \, \hat{g}'(s) \, ds \geq g_0 \int_0^x s \, ds = \dfrac{g_0}{2} \, x^2$, since $\hat{g}(0) = 0$ and $\hat{g}' \geq g_0 \geq 0$ (see assumptions provided in section 2.1). Similarly, for the second term of (2.3.8), using the inequality (2.2.11), we get,

$$\sum_{E\in\mathcal{T}_h} g_0\, S_2^E\left((I-\Pi_p^0)v_h,(I-\Pi_p^0)v_h\right) \geq \mu_* \sum_{E\in\mathcal{T}_h} g_0\|(I-\Pi_p^0)v_h\|_{0,E}^2. \tag{2.3.10}$$

For the last term of (2.3.8), since $\hat{g}(0)=0$,

$$I := \Big|\sum_{E\in\mathcal{T}_h}\tau_E\left(\hat{g}(\Pi_p^0 v_h),\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\right)_E\Big| = \Big|\sum_{E\in\mathcal{T}_h}\tau_E\left(\hat{g}(\Pi_p^0 v_h)-\hat{g}(0),\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\right)_E\Big|.$$

Applying Cauchy-Schwarz inequality and the Lipschitz continuity of $\hat{g}(\cdot)$ with Lipschitz constant L_g, we have,

$$I \leq \sum_{E\in\mathcal{T}_h}\tau_E L_g\|\Pi_p^0 v_h\|_{0,E}\|\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\|_{0,E} \leq \sum_{E\in\mathcal{T}_h}\sqrt{\tau_E}L_g\|\Pi_p^0 v_h\|_{0,E}\sqrt{\tau_E}\|\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\|_{0,E}.$$

Using $\tau_E \leq \dfrac{\sigma+g_0}{4L_g^2}$ and Young's inequality for products, we get,
$I \leq \dfrac{1}{4}\sum_{E\in\mathcal{T}_h}\left((\sigma+g_0)\,\|\Pi_p^0 v_h\|_{0,E}^2+\tau_E\,\|\mathbf{b}\cdot\Pi_{p-1}^0\nabla v_h\|_{0,E}^2\right)$. This implies,

$$\begin{aligned}\sum_{E\in\mathcal{T}_h}\tau_E\left(\hat{g}(\Pi_p^0 v_h),\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\right)_E \;\; &\geq \;\; -\frac{1}{4}\sum_{E\in\mathcal{T}_h}\Big((\sigma+g_0)\,\|\Pi_p^0 v_h\|_{0,E}^2 \\ &\qquad +\tau_E\,\|\mathbf{b}\cdot\Pi_{p-1}^0\nabla v_h\|_{0,E}^2\Big)\end{aligned} \tag{2.3.11}$$

Substituting (2.3.9), (2.3.10) and (2.3.11) into (2.3.8) we get,

$$\begin{aligned} d(v_h,v_h) \;\; &\geq \;\; \frac{g_0}{2}\sum_{E\in\mathcal{T}_h}\|\Pi_p^0 v_h\|_{0,E}^2+\mu_*\sum_{E\in\mathcal{T}_h}g_0\|(I-\Pi_p^0)v_h\|_{0,E}^2 \\ &\quad -\frac{1}{4}\sum_{E\in\mathcal{T}_h}\left((\sigma+g_0)\,\|\Pi_p^0 v_h\|_{0,E}^2+\tau_E\,\|\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\|_{0,E}^2\right).\end{aligned} \tag{2.3.12}$$

Adding (2.3.4), (2.3.5), (2.3.7) and (2.3.12) the desired coercivity result is obtained. □

Lemma 2.2. *Given $u\in H_0^1(\Omega)$ with $(\nabla\cdot(K\nabla u))|_E \in L^2(E)$. Then for all $v_h\in V_h^p$ the following result holds*

$$a(u,v_h)+b(u,v_h)+c(u,v_h) \leq C_I\, N_s(u)\,|||v_h|||, \tag{2.3.13}$$

where, C_I is a positive constant independent of E, h and p and

$$N_s(u) := \Big[(1+\lambda^*)\max_{E\in\mathcal{T}_h}\Big(\frac{K_E+\tau_E\mathbf{b_E}^2}{K_E^\vee}\Big)+\max_{E\in\mathcal{T}_h}\Big(\frac{\mathbf{b_E}}{\sqrt{K_E^\vee\sigma}}\Big)+\max_{E\in\mathcal{T}_h}\Big(\frac{\sqrt{\tau_E}\mathbf{b_E}}{\sqrt{K_E^\vee}}\Big)$$
$$+(1+\mu^*)+\max_{E\in\mathcal{T}_h}\Big(\frac{\sqrt{K_E\tau_E}\mathbf{b_E}}{2K_E^\vee}\Big)\Big]|||u|||+\Big(\sum_{E\in\mathcal{T}_h}\min\Big\{\frac{1}{\tau_E};\frac{\mathbf{b}_E^2}{K_E^\vee}\Big\}\|u\|_{0,E}^2\Big)^{\frac{1}{2}} \quad (2.3.14)$$

Proof. To bound the terms $a(\cdot,\cdot), b(\cdot,\cdot), c(\cdot,\cdot)$, defined in (2.2.2), (2.2.3), (2.2.4), the following inequalities (2.3.15)-(2.3.18) discussed in [12] will be used.

For any $E\in\mathcal{T}_h$,

$$\|\Pi_{p-1}^0\nabla v_h\|_{0,E}\le\|\nabla v_h\|_{0,E}. \quad (2.3.15) \qquad \|\nabla(I-\Pi_p^\nabla)v_h\|_{0,E}\le\|\nabla v_h\|_{0,E}. \quad (2.3.17)$$
$$\|\Pi_p^0 v_h\|_{0,E}\le\|v_h\|_{0,E}. \quad (2.3.16) \qquad \|(I-\Pi_p^0)v_h\|_{0,E}\le\|v_h\|_{0,E}. \quad (2.3.18)$$

Applying (2.2.10), Cauchy-Schwarz inequality, (2.3.15), (2.3.17), H$\ddot{o}$lder's inequality, and from the definition of $|||\cdot|||$ over $a(u,v_h)$, we get,

$$\begin{aligned}
a(u,v_h) &\le \sum_{E\in\mathcal{T}_h}K_E\|\mathbf{\Pi}_{p-1}^0\nabla u\|_{0,E}\|\mathbf{\Pi}_{p-1}^0\nabla v_h\|_{0,E}+\tau_E\mathbf{b}_E^2\|\mathbf{\Pi}_{p-1}^0\nabla u\|_{0,E}\|\mathbf{\Pi}_{p-1}^0\nabla v_h\|_{0,E}\\
&\qquad+\sum_{E\in\mathcal{T}_h}(K_E+\tau_E\mathbf{b}_E^2)\ \lambda^*\|\nabla(I-\Pi_P^\nabla)u\|_{0,E}\,\|\nabla(I-\Pi_P^\nabla)v_h\|_{0,E}\\
&\le \sum_{E\in\mathcal{T}_h}\Big(\frac{K_E}{K_E^\vee}\|\sqrt{K}\nabla u\|_{0,E}\,\|\sqrt{K}\nabla v_h\|_{0,E}+\frac{\tau_E\mathbf{b}_E^2}{K_E^\vee}\|\sqrt{K}\nabla u\|_{0,E}\,\|\sqrt{K}\nabla v_h\|_{0,E}\Big)\\
&\qquad+\sum_{E\in\mathcal{T}_h}\Big(\Big(\frac{K_E+\tau_E\mathbf{b}_E^2}{K_E^\vee}\Big)\lambda^*\|\sqrt{K}\nabla u\|_{0,E}\,\|\sqrt{K}\nabla v_h\|_{0,E}\Big)\\
&\le (1+\lambda^*)\max_{E\in\mathcal{T}_h}\Big(\frac{K_E+\tau_E\mathbf{b}_E^2}{K_E^\vee}\Big)\sum_{E\in\mathcal{T}_h}\|\sqrt{K}\nabla u\|_{0,E}\,\|\sqrt{K}\nabla v_h\|_{0,E}\\
&\le (1+\lambda^*)\max_{E\in\mathcal{T}_h}\Big(\frac{K_E+\tau_E\mathbf{b}_E^2}{K_E^\vee}\Big)|||u|||\,|||v_h|||. \qquad (2.3.19)
\end{aligned}$$

For the term $b(u,v_h)$, using (2.2.11), Cauchy-Schwarz inequality, (2.3.16), (2.3.18) and H$\ddot{o}$lder's inequality, we get,

$$\begin{aligned}
b(u,v_h) &\le \sum_{E\in\mathcal{T}_h}\sigma\Big(\|\Pi_p^0u\|_{0,E}\,\|\Pi_p^0v_h\|_{0,E}+\mu^*\|u-\Pi_p^0u\|_{0,E}\|v_h-\Pi_p^0v_h\|_{0,E}\Big)\\
&\le (1+\mu^*)\Big(\sum_{E\in\mathcal{T}_h}\sigma\,\|u\|_{0,E}^2\Big)^{\frac{1}{2}}\Big(\sum_{E\in\mathcal{T}_h}\sigma\,\|v_h\|_{0,E}^2\Big)^{\frac{1}{2}}.\\
&\le (1+\mu^*)\,|||u|||\,|||v_h|||. \qquad (2.3.20)
\end{aligned}$$

Applying triangle inequality and Cauchy-Schwarz inequality on $c(u, v_h)$, we get,

$$\begin{aligned} c(u, v_h) &\leq \frac{1}{2}\sum_{E\in\mathcal{T}_h} \|\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla u\|_{0,E}\,\|\Pi^0_p v_h\|_{0,E} + \frac{1}{2}\sum_{E\in\mathcal{T}_h} \|\Pi^0_p u\|_{0,E}\,\|\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{0,E} \\ &\quad + \sum_{E\in\mathcal{T}_h} \tau_E\Big(\sigma\,\|\Pi^0_p u\|_{0,E} \;+\; \|-\nabla\cdot K\mathbf{\Pi}^0_{p-1}\nabla u\|_{0,E}\Big)\|\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{0,E}. \\ &:= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV} \end{aligned} \tag{2.3.21}$$

Using (2.3.15), (2.3.16), Hölder's inequality and the definition of $|||\cdot|||$, we get,

$$\mathrm{I} \;\leq\; \sum_{E\in\mathcal{T}_h} \frac{\mathbf{b_E}}{\sqrt{K^\vee_E}}\|\sqrt{K}\nabla u\|_{0,E}\,\frac{\sqrt{\sigma}}{\sqrt{\sigma}}\|v_h\|_{0,E} \leq \max_{E\in\mathcal{T}_h}\left(\frac{\mathbf{b_E}}{\sqrt{K^\vee_E\sigma}}\right)|||u|||\;|||v_h|||. \tag{2.3.22}$$

Next, the second term of (2.3.21) is bounded in two different ways. Using (2.3.15), (2.3.16), we have,

$$\mathrm{II} \leq \sum_{E\in\mathcal{T}_h}\|u\|_{0,E}\left(\frac{\mathbf{b}_E}{\sqrt{K^\vee_E}}\right)\|\sqrt{K}\nabla v_h\|_{0,E} \leq \left(\sum_{E\in\mathcal{T}_h}\left(\frac{\mathbf{b^2_E}}{K^\vee_E}\right)\,\|u\|^2_{0,E}\right)^{\frac{1}{2}}|||v_h|||. \tag{2.3.23}$$

Alternatively, we again bound II as follows,

$$\begin{aligned} \mathrm{II} &\leq \sum_{E\in\mathcal{T}_h}\frac{1}{\sqrt{\tau_E}}\|u\|_{0,E}\sqrt{\tau_E}\frac{\mathbf{b}_E}{\sqrt{K^\vee_E}}\|\sqrt{K}\nabla v_h\|_{0,E} \quad \text{(by (2.3.15), (2.3.16))} \\ &\leq \Big(\sum_{E\in\mathcal{T}_h}\frac{1}{\tau_E}\|u\|^2_{0,E}\Big)^{\frac{1}{2}}\Big(\sum_{E\in\mathcal{T}_h}\Big(\frac{\tau_E\mathbf{b}^2_E}{K^\vee_E}\Big)\|\sqrt{K}\nabla v_h\|^2_{0,E}\Big)^{\frac{1}{2}}\text{(by Hölder's inequality)} \\ &\leq \Big(\sum_{E\in\mathcal{T}_h}\frac{1}{\tau_E}\|u\|^2_{0,E}\Big)^{\frac{1}{2}}\Big(\max_{E\in\mathcal{T}_h}\Big(\frac{(\sigma+g_0)\mathbf{b}^2_E}{L^2_g K^\vee_E}\Big)\Big)^{\frac{1}{2}}|||v_h|||.\Big(\text{using } \tau_E \leq \frac{\sigma+g_0}{L^2_g}\Big) \end{aligned} \tag{2.3.24}$$

Thus, combining the bounds in (2.3.23), (2.3.24), we obtain,

$$\mathrm{II} \leq \Big(\sum_{E\in\mathcal{T}_h}\min\Big\{\frac{1}{\tau_E};\frac{\mathbf{b}^2_E}{K^\vee_E}\Big\}\|u\|^2_{0,E}\Big)^{\frac{1}{2}}C^*_I\,|||v_h|||, \tag{2.3.25}$$

where, $C^*_I = \max\Big\{1,\Big(\max_{E\in\mathcal{T}_h}\Big(\frac{(\sigma+g_0)\mathbf{b}^2_E}{L^2_g K^\vee_E}\Big)\Big)^{\frac{1}{2}}\Big\}$, is a constant independent of h, and E.

Using the assumption $\tau_E \leq \dfrac{1}{4\sigma}$, (2.3.15) and (2.3.16), we get,

$$\mathrm{III} \;\leq\; \sum_{E\in\mathcal{T}_h}\sqrt{\sigma}\,\|\Pi^0_p u\|_E\,\frac{\sqrt{\tau_E}}{2}\,\|\mathbf{b}\cdot\Pi^0_{p-1}\nabla v_h\|_{0,E} \leq \sum_{E\in\mathcal{T}_h}\sqrt{\sigma}\,\|u\|_{0,E}\frac{\sqrt{\tau_E}\mathbf{b}_E}{\sqrt{K^\vee_E}}\,\|\sqrt{K}\nabla v_h\|_{0,E}.$$

Using H$\ddot{o}$lder's inequality and the definition of $|||\cdot|||$, we obtain,

$$\text{III} \leq \left(\max_{E\in\mathcal{T}_h} \frac{\sqrt{\tau_E}\mathbf{b}_E}{\sqrt{K_E^{\vee}}} \right) |||u|||\, |||v_h|||. \tag{2.3.26}$$

Finally we bound the term IV of (2.3.21) as follows,

$$\begin{aligned}
\text{IV} &\leq \sum_{E\in\mathcal{T}_h} \|\sqrt{K}\Pi^0_{p-1}\nabla u\|_{0,E} \frac{\sqrt{\tau_E}}{2}\, \mathbf{b}_E \|\Pi^0_{p-1}\nabla v_h\|_{0,E} \quad \text{(using (2.3.6))} \\
&\leq \sum_{E\in\mathcal{T}_h} \frac{\sqrt{K_E\tau_E}\mathbf{b}_E}{2K_E^{\vee}} \|\sqrt{K}\nabla u\|_{0,E}\, \|\sqrt{K}\nabla v_h\|_{0,E} \quad \text{(using (2.3.15))} \\
&\leq \left(\max_{E\in\mathcal{T}_h} \frac{\sqrt{K_E\tau_E}\mathbf{b}_E}{2K_E^{\vee}} \right) |||u|||\, |||v_h|||. \quad \text{(using Hölder's inequality and } |||\cdot|||)
\end{aligned} \tag{2.3.27}$$

Substituting (2.3.22), (2.3.25), (2.3.26) and (2.3.27) into (2.3.21), we obtain,

$$\begin{aligned}
c(u, v_h) \leq\; & \left[\max_{E\in\mathcal{T}_h} \left(\frac{\mathbf{b_E}}{\sigma\sqrt{K_E^{\vee}}} \right) + \max_{E\in\mathcal{T}_h} \left(\frac{\sqrt{\tau_E}\mathbf{b_E}}{\sqrt{K_E^{\vee}}} \right) + \max_{E\in\mathcal{T}_h} \left(\frac{\sqrt{K_E\tau_E}\mathbf{b_E}}{2K_E^{\vee}} \right) \right] |||u|||\, |||v_h||| \\
& + \Big(\sum_{E\in\mathcal{T}_h} \min \Big\{ \frac{1}{\tau_E}; \frac{\mathbf{b}_E^2}{K_E^{\vee}} \Big\} \|u\|^2_{0,E} \Big)^{\frac{1}{2}} C_I^* \, |||v_h|||.
\end{aligned} \tag{2.3.28}$$

Now, adding (2.3.19), (2.3.20) and (2.3.28), and letting $C_I = \max(1, C_I^*)$ we get the desired result. □

The following proposition (see [50]) is useful for showing the existence and uniqueness of the discrete solution for the problem (2.2.8).

Proposition 2.1. *Let H be a finite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. Let $P : H \to H$ be a strongly monotone and Lipschitz continuous operator. Then $P(u) = f$ has a unique solution for all $f \in H$.*

Remark 2.1. Let us define the inner product $\langle\cdot,\cdot\rangle_M$ on V_h^p by $\langle w_h, v_h\rangle_M = \sum_{E\in\mathcal{T}_h} (\nabla w_h, \nabla v_h)_E$ $\forall w_h, v_h \in V_h^p$, and denote, the induced norm by $\|\cdot\|_M$. We note that V_h^p with inner product $\langle\cdot,\cdot\rangle_M$ is a finite dimensional Hilbert space. The norms $|||\cdot|||$ and $\|\cdot\|_M$ are equivalent on V_h^p. That is, there exists $k_1 > 0$ and $k_2 > 0$, such that,

$$k_1\|v_h\|_M \leq |||v_h||| \leq k_2\|v_h\|_M \quad \forall v_h \in V_h^p. \tag{2.3.29}$$

Theorem 2.1 (Well-posedness)**.** *Let the assumptions on the problem* (2.1.1) *be satisfied. Then the VEM-SUPG scheme* (2.2.8) *has a unique solution* $u_h \in V_h^p$.

Proof. In order to use Proposition 2.1, we first rewrite (2.2.8) in the operator form on V_h^p with inner product $\langle \cdot, \cdot \rangle_M$ and norm $\| \cdot \|_M$.
For each $y_h \in V_h^p$, let us define the operator $T_{y_h} : V_h^p \to \mathbb{R}$ by $T_{y_h}(z_h) = A_{vsg}(y_h, z_h)$. Note that for each $y_h \in V_h^p$, the corresponding T_{y_h} is a bounded linear functional on V_h^p. Now, using Riesz representation theorem, there exists a unique $q_h \in V_h^p$ such that $T_{y_h}(z_h) = \langle q_h, z_h \rangle_M, \ \forall z_h \in V_h^p$.
The correspondence $y_h \to q_h$, defines a mapping $M : V_h^p \to V_h^p$ such that

$$\langle M(y_h), z_h \rangle_M = A_{vsg}(y_h, z_h) \quad \forall z_h \in V_h^p. \tag{2.3.30}$$

Consider F_{vsg} in (2.2.6) for a fixed $f \in L^2(\Omega)$. We have

$$|F_{vsg}(z_h)| \leq (C_p + \max_{E \in \mathcal{T}_h} \mathbf{b}_E \tau_E) \, \|f\|_{0,\Omega} \|z_h\|_M \ \ \forall z_h \in V_h^p,$$

where C_p denotes the Poincare constant. Thus for fixed $f \in L^2(\Omega)$, F_{vsg} is a bounded linear operator on V_h^p. Again by Riesz representation theorem there exists a unique $f_{vsg} \in V_h^p$ such that

$$F_{vsg}(z_h) = \langle f_{vsg}, \, z_h \rangle_M \quad \forall z_h \in V_h^p. \tag{2.3.31}$$

Hence using (2.3.30) and (2.3.31), we note that the scheme (2.2.8) is equivalent to the following operator form : Find $u_h \in V_h^p$ such that

$$M(u_h) = f_{vsg}. \tag{2.3.32}$$

Now we show that M is strongly monotone. Consider $v_h, \, w_h \in V_h^p$, let $\phi := v_h - w_h$ and $M_\phi = M(v_h) - M(w_h)$. We estimate $\langle M_\phi, \, \phi \rangle_M := \langle M(v_h) - M(w_h), \, v_h - w_h \rangle_M$. We have, $\langle M_\phi, \, \phi \rangle_M = A_{vsg}(v_h, v_h - w_h) - A_{vsg}(w_h, v_h - w_h) = a(\phi, \phi) + b(\phi, \phi) + c(\phi, \phi) + d(v_h, \phi) - d(w_h, \phi)$. Therefore, we have

$$\begin{aligned}\langle M_\phi, \, \phi \rangle_M &= a(\phi, \phi) + b(\phi, \phi) + c(\phi, \phi) + \sum_{E \in \mathcal{T}_h} \big(\hat{g}(\Pi_p^0 v_h) - \hat{g}(\Pi_p^0 w_h), \Pi_p^0 \phi \big)_E \\ &\quad + \sum_{E \in \mathcal{T}_h} g_0 \, S_2^E \big((I - \Pi_p^0)\phi, (I - \Pi_p^0)\phi \big) + \sum_{E \in \mathcal{T}_h} \tau_E \Big(\hat{g}(\Pi_p^0 v_h) - \hat{g}(\Pi_p^0 w_h), \mathbf{b} \cdot \mathbf{\Pi^0_{p-1}} \nabla \phi \Big)_E \\ &= a(\phi, \phi) + b(\phi, \phi) + c(\phi, \phi) + I + II + III. \end{aligned} \tag{2.3.33}$$

Term I is bounded by using mean value theorem on $\hat{g}(\cdot)$ and assumption (A1.c). Using (2.2.11) we bound II. Thus we obtain,

$$\mathrm{I} \geq \frac{g_0}{2} \sum_{E\in\mathcal{T}_h} \|\Pi_p^0\phi\|_{0,E}^2. \tag{2.3.34}$$

$$\mathrm{II} \geq g_0\,\mu_* \sum_{E\in\mathcal{T}_h} \|(I-\Pi_p^0)\phi\|_{0,E}^2. \tag{2.3.35}$$

Applying Cauchy-Schwarz inequality and the Lipschitz continuity of $\hat{g}(\cdot)$ we have,

$$\begin{aligned}
|\mathrm{III}| &\leq \sum_{E\in\mathcal{T}_h} \tau_E\, L_g\, \|\Pi_p^0\phi\|_{0,E}\|\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0\nabla\phi\|_{0,E} \\
&\leq \sum_{E\in\mathcal{T}_h} \sqrt{\tau_E}\, L_g\, \|\Pi_p^0\phi\|_{0,E}\sqrt{\tau_E}\, \|\mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0\nabla\phi\|_{0,E}.
\end{aligned}$$

Similar to (2.3.11), we get,

$$\mathrm{III} \;\geq\; -\frac{1}{4}\sum_{E\in\mathcal{T}_h} \left((\sigma+g_0)\,\|\Pi_p^0\phi\|_{0,E}^2 + \tau_E\,\|\mathbf{b}\cdot\Pi_{p-1}^0\nabla\phi\|_{0,E}^2\right). \tag{2.3.36}$$

Letting $v_h = \phi$ in (2.3.4), (2.3.5) and (2.3.7), we obtain bounds for $a(\phi,\phi)$, $b(\phi,\phi)$ and $c(\phi,\phi)$ respectively. Therefore substituting (2.3.4), (2.3.5), (2.3.7) and (2.3.34)-(2.3.36) into (2.3.33), and using Lemma 2.1 and (2.3.29), we get for any $v_h, w_h \in V_h^p$,

$$\langle M(v_h) - M(w_h),\, v_h - w_h\rangle_M \geq \theta\,\|\!|v_h - w_h|\!\|^2 \geq \theta\, k_1^2 \|v_h - w_h\|_M^2. \tag{2.3.37}$$

Next we prove Lipschitz continuity of M. Consider,

$$\begin{aligned}
\langle M_\phi, M_\phi\rangle_M &= \langle M(v_h), M_\phi\rangle_M - \langle M(w_h), M_\phi\rangle_M = A_{vsg}(v_h, M_\phi) - A_{vsg}(w_h, M_\phi) \\
&= a(\phi, M_\phi) + b(\phi, M_\phi) + c(\phi, M_\phi) + \sum_{E\in\mathcal{T}_h} \left(\hat{g}(\Pi_p^0 v_h) - \hat{g}(\Pi_p^0 w_h), \Pi_p^0 M_\phi\right)_E \\
&\quad + \sum_{E\in\mathcal{T}_h} g_0\, S_2^E\left((I-\Pi_p^0)\phi, (I-\Pi_p^0)M_\phi\right) \\
&\quad + \sum_{E\in\mathcal{T}_h} \tau_E\Big(\hat{g}(\Pi_p^0 v_h) - \hat{g}(\Pi_p^0 w_h), \mathbf{b}\cdot\mathbf{\Pi_{p-1}^0}\nabla M_\phi\Big)_E \\
&= a(\phi, M_\phi) + b(\phi, M_\phi) + c(\phi, M_\phi) + \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{2.3.38}$$

In (2.3.14), bounding the L^2 norm in $N_s(\cdot)$ by $\|\!|\cdot|\!\|$ and using (2.3.29) we get,

$$N_s(\phi) \leq C_{Ns}\|\phi\|_M,$$

where $C_{Ns} > 0$ is dependent on K_E, $\mathbf{b}_E$, σ, τ_E, μ^*, λ^* and k_2.

Hence Lemma 2.2 and (2.3.29) implies,

$$a(\phi, M_\phi) + b(\phi, M_\phi) + c(\phi, M_\phi) \leq \mathcal{C}\|\phi\|_M \|M_\phi\|_M, \tag{2.3.39}$$

where $\mathcal{C} = C_I C_{Ns} k_2$.

Next, using Cauchy-Schwarz inequality, Lipschitz continuity of $\hat{g}(\cdot)$, (2.3.16), H$\ddot{o}$lder's inequality, and then, Poincar$\acute{e}$ inequality, we get,

$$\begin{aligned} \mathrm{I} &\leq \sum_{E\in\mathcal{T}_h} L_g \|\Pi_p^0 \phi\|_{0,E} \|\Pi_p^0 M_\phi\|_{0,E} \\ &\leq L_g \Big(\sum_{E\in\mathcal{T}_h} \|\phi\|_{0,E}^2\Big)^{\frac{1}{2}} \Big(\textstyle\sum_{E\in\mathcal{T}_h} \|M_\phi\|_{0,E}^2\Big)^{\frac{1}{2}} \leq L_g\, C_P^2\, \|\phi\|_M \|M_\phi\|_M. \end{aligned} \tag{2.3.40}$$

Using (2.2.11), (2.3.18), H$\ddot{o}$lder's inequality, and then, Poincar$\acute{e}$ inequality, we get,

$$\mathrm{II} \leq g_0\, \mu^*\, C_P^2\, \|\phi\|_M \|M_\phi\|_M. \tag{2.3.41}$$

Using $\tau_E \leq \dfrac{\sigma + g_0}{4L_g^2}$, and estimating in a similar way, we obtain,

$$\begin{aligned} \mathrm{III} &\leq \sum_{E\in\mathcal{T}_h} \frac{\mathbf{b}_E(\sigma + g_0)}{4L_g} \|\phi\|_{0,E}\, \|\nabla M_\phi\|_{0,E} \\ &\leq (\max_{E\in\mathcal{T}_h} \mathbf{b}_E) \frac{(\sigma + g_0)}{4L_g} C_P \Big(\sum_{E\in\mathcal{T}_h} \|\nabla\phi\|_{0,E}^2\Big)^{\frac{1}{2}} \Big(\sum_{E\in\mathcal{T}_h} \|\nabla M_\phi\|_{0,E}^2\Big)^{\frac{1}{2}} \\ &\leq (\max_{E\in\mathcal{T}_h} \mathbf{b}_E) \frac{(\sigma + g_0)}{4L_g} C_P \|\phi\|_M \|M_\phi\|_M. \end{aligned} \tag{2.3.42}$$

Substituting the equations (2.3.39)-(2.3.42) into (2.3.38), implies that there exists constant $C > 0$, such that $\|M_\phi\|_M^2 \leq C\|\phi\|_M \|M_\phi\|_M$. That is,

$$\|M(v_h) - M(w_h)\|_M \;\;\leq\;\; C\|v_h - w_h\|_M \quad \forall v_h, w_h \in V_h^p. \tag{2.3.43}$$

Thus (2.3.37) and (2.3.43) imply that M is a strongly monotone and Lipschitz continuous operator on V_h^p respectively. Now using Proposition 2.1 (thanks to Remark 2.1), we get that (2.3.32) has a unique solution. This implies that the scheme (2.2.8) has a unique solution. □

2.3.2 Convergence results

In this section we will prove the rate of convergence results first by deriving a lemma on the $|||u - u_h|||$ with respect to the continuity coefficient N_s, then we use the lemma to prove a theorem that estimates the rate of convergence.

Lemma 2.3. *Let the stabilization parameter* τ_E *satisfies*

$$0 \le \tau_E \le \frac{\theta}{256} \min\Big\{\frac{h_E^2}{p^4\,\mu_{inv}^2 K_E}; \frac{1}{\sigma}; \frac{\sigma + g_0}{L_g^2}\Big\},$$

where θ *is as defined in Lemma 2.1. Further let* $u \in H_0^1(\Omega)$ *satisfy* (2.1.1) *and the assumption* $(\nabla\cdot(K\nabla u))\,|_E \in L^2(E)$ *for all* $E \in \mathcal{T}_h$ *and if* σ *is chosen such that,* $(\sigma + g_0)\theta > 12(L_g + g_0\mu^*)$*, then for sufficiently small* h*,*

$$|||u - u_h||| \le \widehat{C} \inf_{v_h \in V_h^p} N_s(u - v_h), \tag{2.3.44}$$

where $\widehat{C}$ *depends, in particular, on* σ*,* g_0*,* μ^**,* θ *and* L_g*.*

Proof. Let $u_h \in V_h^p$ be the discrete solution satisfying the VEM discretization (2.2.8). For arbitrary $v_h \in V_h^p$, let $\phi := u - v_h$, $\psi := u_h - v_h$ and $e := u - u_h = \phi - \psi$.
First, we find a bound for $|||\psi|||$, in terms of $|||e|||$ and $|||\phi|||$. Note that, both u and u_h satisfy (2.2.8). Therefore, $A_{vsg}(u, w_h) - A_{vsg}(u_h, w_h) = 0 \quad \forall w_h \in V_h^p$. This implies,

$$a(e, w_h) + b(e, w_h) + c(e, w_h) + d(u, w_h) - d(u_h, w_h) = 0 \;\; \forall w_h \in V_h^p. \tag{2.3.45}$$

Hence for $\psi \in V_h^p$, using Lemma 2.1 and (2.3.45), we get,

$$\begin{aligned}
\theta|||\psi|||^2 &\le A_{vsg}(\psi,\psi) = a(\phi - e,\psi) + b(\phi - e,\psi) + c(\phi - e,\psi) + d(\psi,\psi)\\
&\le a(\phi,\psi) + b(\phi,\psi) + c(\phi,\psi) - (a(e,\psi) + b(e,\psi) + c(e,\psi)) + d(\psi,\psi)\\
&\le a(\phi,\psi) + b(\phi,\psi) + c(\phi,\psi) + \Big(d(u,\psi) - d(u_h,\psi)\Big) + d(\psi,\psi)\\
&\le a(\phi,\psi) + b(\phi,\psi) + c(\phi,\psi) + \sum_{E\in\mathcal{T}_h} g_0 S_2^E((I - \Pi_p^0)\,e, (I - \Pi_p^0)\,\psi)\\
&\quad + \sum_{E\in\mathcal{T}_h} \big(\hat{g}(\Pi_p^0 u) - \hat{g}(\Pi_p^0 u_h), \Pi_p^0\psi\big)_E\\
&\quad + \sum_{E\in\mathcal{T}_h} \tau_E \big(\hat{g}(\Pi_p^0 u) - \hat{g}(\Pi_p^0 u_h), \mathbf{b}\cdot\mathbf{\Pi}_{p-1}^0\nabla\psi\big)_E + d(\psi,\psi)\\
&\le a(\phi,\psi) + b(\phi,\psi) + c(\phi,\psi) + \mathrm{I} + \mathrm{II} + \mathrm{III} + d(\psi,\psi).
\end{aligned} \tag{2.3.46}$$

Using Lemma 2.2 and inequality $\dfrac{m}{\sqrt{\alpha}}\sqrt{\alpha}n \leq \dfrac{m^2}{\alpha} + \alpha n^2$ (choosing $\alpha = \dfrac{\theta}{2}$), we obtain,

$$a(\phi,\psi) + b(\phi,\psi) + c(\phi,\psi) \leq C_I N_s(\phi) |||\psi||| \leq \frac{2}{\theta} C_I^2 (N_s(\phi))^2 + \frac{\theta}{2} |||\psi|||^2. \qquad (2.3.47)$$

Applying (2.2.11), using (2.3.18), Hölder's inequality and then Young's inequality for products, we obtain,

$$\begin{aligned}
\mathrm{I} &\leq \sum_{E\in\mathcal{T}_h} g_0\,\mu^*\,\|e\|_{0,E}\|\psi\|_{0,E} \\
&\leq \frac{g_0\,\mu^*}{\sigma+g_0}\Big(\sum_{E\in\mathcal{T}_h}(\sigma+g_0)\|u-u_h\|_{0,E}^2\Big)^{\frac{1}{2}}\Bigg(\sum_{E\in\mathcal{T}_h}(\sigma+g_0)\|\psi\|_{0,E}^2\Big)^{\frac{1}{2}} \\
&\leq \sqrt{\frac{g_0\,\mu^*}{\sigma+g_0}}\,|||u-u_h|||\,\sqrt{\frac{g_0\,\mu^*}{\sigma+g_0}}\,|||\psi||| \\
&\leq \frac{g_0\,\mu^*}{\sigma+g_0}|||u-u_h|||^2 + \frac{g_0\,\mu^*}{\sigma+g_0}|||\psi|||^2. \qquad (2.3.48)
\end{aligned}$$

Using Lipschitz continuity of $\hat{g}$, (2.3.16), Hölder's inequality, then Young's inequality for products, we get,

$$\begin{aligned}
\mathrm{II} &\leq \sum_{E\in\mathcal{T}_h} L_g\|u-u_h\|_{0,E}\|\psi\|_{0,E} \\
&\leq \frac{L_g}{\sigma+g_0}\Big(\sum_{E\in\mathcal{T}_h}(\sigma+g_0)\|u-u_h\|_{0,E}^2\Big)^{\frac{1}{2}}\Big(\sum_{E\in\mathcal{T}_h}(\sigma+g_0)\|\psi\|_{0,E}^2\Big)^{\frac{1}{2}} \\
&\leq \sqrt{\frac{L_g}{(\sigma+g_0)}}\,|||u-u_h|||\,\sqrt{\frac{L_g}{(\sigma+g_0)}}\,|||\psi||| \\
&\leq \frac{L_g}{(\sigma+g_0)}|||u-u_h|||^2 + \frac{L_g}{(\sigma+g_0)}|||\psi|||^2. \qquad (2.3.49)
\end{aligned}$$

Again using Cauchy-Schwarz inequality, Lipschitz continuity of $\hat{g}$, (2.3.15), and the assumption $\tau_E \leq \dfrac{\theta}{256}\dfrac{h_E^2}{p^4\mu_{inv}^2 K_E}$ we have,

$$\begin{aligned}
\mathrm{III} &\leq \sum_{E\in\mathcal{T}_h}\sqrt{\tau_E}L_g\|u-u_h\|_{0,E}\sqrt{\tau_E}\mathbf{b}_E\|\nabla\psi\|_{0,E} \\
&\leq \sum_{E\in\mathcal{T}_h}\sqrt{\tau_E}L_g\|u-u_h\|_{0,E}\sqrt{\frac{\theta}{256}}\frac{\mathbf{b}_E h_E}{p^2\mu_{inv}\sqrt{K_E}}\|\nabla\psi\|_{0,E} \\
&\leq \sqrt{\frac{\theta}{256}}\sum_{E\in\mathcal{T}_h}\sqrt{\tau_E}L_g\|u-u_h\|_{0,E}\frac{\mathbf{b}_E h_E}{\mu_{inv}K_E^{\vee}}\|\sqrt{K}\nabla\psi\|_{0,E}. \quad \Big(\text{as } p\geq 1,\ \tfrac{1}{K_E}\leq\tfrac{1}{K_E^{\vee}}\Big)
\end{aligned}$$

For $h_E \leq \dfrac{\mu_{inv} K_E^{\vee}}{\mathbf{b}_E}$, and using $\tau_E \leq \dfrac{\theta}{256}\dfrac{\sigma+g_0}{L_g^2}$, Hölder's inequality, the definition of $|||\cdot|||$, and then using Young's inequality for products, we obtain,

$$
\begin{aligned}
\mathrm{III} \;&\leq\; \frac{\theta}{256}\left((\sigma+g_0)\sum_{E\in\mathcal{T}_h}\|u-u_h\|_{0,E}^2\right)^{\frac{1}{2}} |||\psi||| \\
&\leq\; \sqrt{\frac{\theta}{256}}|||u-u_h|||\,\sqrt{\frac{\theta}{256}}|||\psi||| \leq \frac{\theta}{256}|||u-u_h|||^2 + \frac{\theta}{256}|||\psi|||^2. \qquad (2.3.50)
\end{aligned}
$$

Similar to the inequalities (2.3.48) - (2.3.50), repectively, we obtain,

$$
d(\psi,\psi) \leq \frac{g_0\,\mu^*}{\sigma+g_0}|||\psi|||^2 + \frac{L_g}{\sigma+g_0}|||\psi|||^2 + \frac{\theta}{256}|||\psi|||^2. \qquad (2.3.51)
$$

Now, substituting the results obtained in equations (2.3.47)-(2.3.51), into (2.3.46), combining similar terms and simplifying their coefficients, we get,

$$
|||\psi|||^2 \leq \beta_1\left\{\frac{2}{\theta}C_I^2(N_s(\phi))^2 + \left(\frac{L_g+g_0\mu^*}{\sigma+g_0}+\frac{\theta}{256}\right)|||u-u_h|||^2\right\}, \qquad (2.3.52)
$$

where, $\beta_1 = \dfrac{128(\sigma+g_0)}{63(\sigma+g_0)\theta - 128(2L_g+2g_0\mu^*)} > 0$ (using $(\sigma+g_0)\theta > 12(L_g+g_0\mu^*)$).
Substituting (2.3.52) in the inequality $|||u-u_h|||^2 \leq 2|||\phi|||^2 + 2|||\psi|||^2$ we get,

$$
|||u-u_h|||^2 \leq 2|||\phi|||^2 + 2\beta_1\Big\{\frac{2}{\theta}C_I^2(N_s(\phi))^2 + \Big(\frac{L_g+g_0\mu^*}{\sigma+g_0}+\frac{\theta}{256}\Big)|||u-u_h|||^2\Big\}.
$$

Rearranging and simplifying the coefficients, we get,

$$
|||u-u_h|||^2 \leq 2\beta_2|||\phi|||^2 + \beta_1\beta_2\frac{4}{\theta}C_I^2(N_s(\phi))^2, \qquad (2.3.53)
$$

where, $\beta_2 = \dfrac{63(\sigma+g_0)\theta - 128(2L_g+2g_0\mu^*)}{62(\sigma+g_0)\theta - 128(4L_g+4g_0\mu^*)} > 0$ (since $(\sigma+g_0)\theta > 12(L_g+g_0\mu^*)$).
Simplifying the right-hand side (2.3.53) we have,

$$
|||u-u_h|||^2 \;\leq\; R\frac{8}{\theta}\beta_1\beta_2 C_I^2(N_s(\phi))^2,
$$

where, $R := \max(1, 1/J)$ and $J := (4/\theta)\,\beta_1 C_I^2 C_1^2$. Thus, for $v_h \in V_h^p$, $|||u-u_h|||^2 \leq \widehat{C}(N_s(u-v_h))^2$, where, $\widehat{C} = R\frac{8}{\theta}\beta_1\beta_2 C_I^2$. This implies the required assertion (2.3.44). □

Hereafter, C denotes a generic positive constant independent of h_E, p and h, that has

different meaning at different occurrences. In some instances, C may depend on the coefficients of problem (2.1.1).
The following proposition is proved in Lemma 4.2 in [51].

Proposition 2.2. *Consider $E \in \mathcal{T}_h$ satisfying the assumptions (i),(ii) given in section 1.3 and let $u \in H^{k+1}(E)$. Then for each $p \in \mathbb{N}$ there exists a projection operator I^E that maps u onto the polynomial space $\mathbb{P}_p(E)$ such that $0 \le l \le k+1$, $\mu = min(p,k)$,*

$$|u - I^E u|_{l,E} \le C \frac{h_E^{\mu+1-l}}{p^{k+1-l}} \|u\|_{k+1,E}. \tag{2.3.54}$$

Let $\widetilde{\mathcal{T}}_h$ denote the triangulation refinement of $\mathcal{T}_h$: for each $E \in \mathcal{T}_h$, the triangles are formed by joining the vertices of E to the centre of the corresponding disc D_γ in assumption (i) of the mesh regularity condition stated in section 1.3. Denote by $\mathcal{P}_h^p(\widetilde{\mathcal{T}}_h)$ the space of continuous piecewise polynomials of degree $p \in \mathbf{N}$ over $\widetilde{\mathcal{T}}_h$. For $T \in \widetilde{\mathcal{T}}_h$ we denote $\tilde{T}$, to be either T itself or union of T and its immediate neighbours.

Proposition 2.3. *For every $u \in H^{k+1}(\tilde{T})$ there exists $\mathcal{I}u \in \mathcal{P}_h^p(\widetilde{\mathcal{T}}_h)$, (see hypothesis (4.6) in [52]) such that, $\mu = min(p,k)$,*

$$\|u - \mathcal{I}u\|_{0,T} + \frac{h_T}{p} |u - \mathcal{I}u|_{1,T} \le C \frac{h_T^{\mu+1}}{p^{k+1}} \|u\|_{k+1,\tilde{T}}. \tag{2.3.55}$$

Now, we prove a lemma to obtain an estimate involving h and p, for the virtual element interpolation term, following the procedure given in [53].

Lemma 2.4. *Let $E \in \mathcal{T}_h$ be a convex polygon satisfying the assumptions (i),(ii) given in section 1.3. Then, for $u \in H_0^1(\Omega)$ with $u|_E \in H^{k+1}(E)$, $k \in \mathbb{N}$, there exists $u_I \in V_h^p$ satisfying the following,*

$$\|u - u_I\|_{0,E} + \frac{h_E}{p} |u - u_I|_{1,E} \le C \frac{h_E^{\mu+1}}{p^{k+1}} \|u\|_{k+1,E}, \tag{2.3.56}$$

where $\mu = min(p,k)$.

Proof. For each $E \in \mathcal{T}_h$ and $\mathcal{I}u \in \mathcal{P}_h^p(\widehat{\mathcal{T}}_h)$ satisfying (2.3.55) , it is possible to define $u_I|_E \in V_E^p$ (see [51] and [53]) as the solution of the following problem : Find $u_I|_E \in V_E^p$ such that

$$u_I = \mathcal{I}u \quad \text{on } \partial E, \text{ and } (\nabla u_I, \nabla v_h)_E = (\nabla \mathcal{I}u, \nabla v_h)_E \quad \forall v_h \in V_E^p \cap H_0^1(E). \tag{2.3.57}$$

Moreover, since $u_I \in H^1(\Omega)$ we note that $u_I \in V_h^p$. From (2.3.57) we have (see [53]),

$$|\mathcal{I}u - u_I|_{1,E} = \inf \Big\{ |\mathcal{I}u - z_h|_{1,E} \,:\, z_h \in V_E^p \text{ and } z_h = \mathcal{I}u \text{ on } \partial E \Big\}.$$

Therefore,

$$|u - u_I|_{1,E} \le |u - \mathcal{I}u|_{1,E} + |\mathcal{I}u - u_I|_{1,E} \le |u - \mathcal{I}u|_{1,E} + |\mathcal{I}u - \widehat{u}|_{1,E}, \quad (2.3.58)$$

where, $\widehat{u}$ is such that $\widehat{u} \in V_p^E$ is a solution of the problem (see [51]),

$$\begin{aligned} \Delta \widehat{u} &= \Delta I^E u \text{ in } E \\ \widehat{u} &= \mathcal{I}u \quad \text{on } \partial E, \end{aligned}$$

where $I^E u$ is as in Proposition 2.2 satisfying (2.3.54).

Since $(\widehat{u} - I^E u)$ is harmonic in E we get,

$$|\widehat{u} - I^E u|_{1,E} \le |I^E u - \mathcal{I}u|_{1,E}. \quad (2.3.59)$$

Substituting (2.3.59) into (2.3.58) we get,

$$\begin{aligned} |u - u_I|_{1,E} &\le |u - \mathcal{I}u|_{1,E} + |\mathcal{I}u - \widehat{u}|_{1,E} \\ &\le |u - \mathcal{I}u|_{1,E} + |\mathcal{I}u - I^E u|_{1,E} + |I^E u - \widehat{u}|_{1,E} \\ &\le |u - \mathcal{I}u|_{1,E} + |\mathcal{I}u - I^E u|_{1,E} + |I^E u - \mathcal{I}u|_{1,E} \\ &\le 3|u - \mathcal{I}u|_{1,E} + 2|u - I^E u|_{1,E}. \end{aligned} \quad (2.3.60)$$

Applying the results (2.3.54)-(2.3.55) in (2.3.60), and $\mu = \min(p, k)$, we get,

$$|u - u_I|_{1,E} \le C \frac{h_E^\mu}{p^k} \|u\|_{k+1,E}. \quad (2.3.61)$$

To bound the term $\|u - u_I\|_{0,E}$, we consider the following auxiliary problem : Find $\varphi \in H_0^1(E)$ such that

$$(\nabla \varphi, \nabla v)_E = (\mathcal{I}u - u_I, v)_E \quad \forall v \in H_0^1(E). \quad (2.3.62)$$

Using $u_I - \mathcal{I}u = 0$ on ∂E, and (2.3.57) we get,

$$\begin{aligned}\|u_I - \mathcal{I}u\|_{0,E}^2 = (\nabla\varphi, \nabla(u_I - \mathcal{I}u))_E &= (\nabla(\varphi - I^E\varphi), \nabla(u_I - \mathcal{I}u))_E \\ &\le |\varphi - I^E\varphi|_{1,E}\, |u_I - \mathcal{I}u|_{1,E}. \quad (2.3.63)\end{aligned}$$

where $I^E\varphi \in V_E^p \cap H_0^1(E)$ satisfies (2.3.54) giving the estimate,

$$|\varphi - I^E\varphi|_{1,E} \le C\,\frac{h_E}{p}\|\varphi\|_{2,E}. \quad (2.3.64)$$

Substituting (2.3.64) into (2.3.63) and noting that $\|\varphi\|_{2,E} \le C\,\|u_I - \mathcal{I}u\|_{0,E}$, we get,

$$\|u_I - \mathcal{I}u\|_{0,E} \le C\,\frac{h_E}{p}|u_I - \mathcal{I}u|_{1,E}. \quad (2.3.65)$$

Now, using (2.3.65), applying the results (2.3.55) and (2.3.61) with $\mu = \min(p,k)$, we get,

$$\begin{aligned}\|u - u_I\|_{0,E} &\le \|u - \mathcal{I}u\|_{0,E} + \|\mathcal{I}u - u_I\|_{0,E} \\ &\le \|u - \mathcal{I}u\|_{0,E} + C\,\frac{h_E}{p}|u_I - \mathcal{I}u|_{1,E} \\ &\le \|u - \mathcal{I}u\|_{0,E} + C\,\frac{h_E}{p}|u_I - u|_{1,E} + C\,\frac{h_E}{p}|u - \mathcal{I}u|_{1,E}. \\ &\le C\,\frac{h_E^{\mu+1}}{p^{k+1}}\|u\|_{k+1,E}. \quad (2.3.66)\end{aligned}$$

□

Next, using auxillary lemma 2.3 and hp virtual interpolation estimate in (2.3.56), we derive a convergence estimate with respect to $|||\cdot|||$ for the solution of the SUPG stabilized VEM scheme (2.2.8).

Theorem 2.2. *Let assumptions on τ_E, σ from the Lemma 2.3 be satisfied. Let $u_h \in V_h^p$ satisfy problem* (2.2.8) *and let $u \in H_0^1(\Omega)$ be the solution of the problem* (2.1.1) *with $u \in H^{s+1}(E)$, $p \ge s > 1$, and E is convex, $\forall E \in \mathcal{T}_h$. For sufficiently small h, the following estimate holds*

$$\begin{aligned}|||u - u_h|||^2 \le\ & C \sum_{E\in\mathcal{T}_h} \frac{h_E^{2s}}{p^{2s}}\,\|u\|_{s+1,E}^2\Big(K_E + \frac{(\sigma+g_0)h_E^2}{p^2} \\ & + \tau_E\,\mathbf{b}_E^2 + \min\Big\{\frac{1}{\tau_E}; \frac{\mathbf{b}_E^2}{K_E^{\vee}}\Big\}\frac{h_E^2}{p^2}\Big), \quad (2.3.67)\end{aligned}$$

Proof. Consider the interpolant $u_I \in V_h^p$. Then Lemma 2.3 implies,

$$\begin{aligned}
|||u-u_h|||^2 &\leq C\, N_s(u-u_I)^2 \\
&\leq C\Big[|||u-u_I||| + \Big(\sum_{E\in\mathcal{T}_h} \min\Big\{\frac{1}{\tau_E}; \frac{\mathbf{b}_E^2}{K_E^{\vee}}\Big\}\|u-u_I\|_{0,E}^2\Big)^{\frac{1}{2}}\Big]^2 \\
&\leq C\Big(|||u-u_I|||^2 + \sum_{E\in\mathcal{T}_h} \min\Big\{\frac{1}{\tau_E}; \frac{\mathbf{b}_E^2}{K_E^{\vee}}\Big\}\|u-u_I\|_{0,E}^2\Big) \\
&\leq C\Big[\sum_{E\in\mathcal{T}_h}\Big(\|\sqrt{K}\,\nabla(u-u_I)\|_{0,E}^2 + (\sigma+g_0)\,\|u-u_I\|_{0,E}^2 \\
&\qquad +\tau_E\,\|\mathbf{b}.\nabla(u-u_I)\|_{0,E}^2\Big) + \sum_{E\in\mathcal{T}_h} \min\Big\{\frac{1}{\tau_E}; \frac{\mathbf{b}_E^2}{K_E^{\vee}}\Big\}\|u-u_I\|_{0,E}^2\Big] \\
&\leq C\Big[\sum_{E\in\mathcal{T}_h}\Big(K_E|u-u_I|_{1,E}^2 + (\sigma+g_0)\,\|u-u_I\|_{0,E}^2 + \tau_E\,\mathbf{b}_E^2|u-u_I|_{1,E}^2\Big) \\
&\qquad + \sum_{E\in\mathcal{T}_h} \min\Big\{\frac{1}{\tau_E}; \frac{\mathbf{b}_E^2}{K_E^{\vee}}\Big\}\|u-u_I\|_{0,E}^2\Big].
\end{aligned}$$

Now using (2.3.56) from Lemma 2.4, the desired result (2.3.67) is obtained. □

In the following, we will present a suitable choice for τ_E (proof is similar to [47], Corollary 2.1.)

Corollary 2.1. *Using the assumptions of Theorem 2.2 along with* $\frac{1}{\tau_E} \leq \frac{\mathbf{b}_E^2}{K_E^{\vee}}$ *and considering the following choice for* τ_E *in* (2.3.67),

$$\tau_E \sim \min\Big\{\frac{h_E}{p\mathbf{b}_E};\ \frac{h_E^2}{p^4\,\mu_{inv}^2 K_E}; \frac{1}{\sigma+g_0}; \frac{\sigma+g_0}{L_g^2}\Big\}.$$

Let us denote $Pe_E := \frac{h_E\,\mathbf{b}_E}{p\,K_E}$, $\Upsilon_E^{(t)} := \frac{(\sigma+g_0)h_E^2}{p^2\,K_E}$, $\Upsilon_E^{(r)} := \frac{L_g^2\,h_E^2}{(\sigma+g_0)\,p^2\,K_E}$. *Then we obtain,*

$$|||u-u_h|||^2 \leq C \sum_{E\in\mathcal{T}_h} \frac{h_E^{2s}}{p^{2s}}\,\mathcal{R}_E^{opt}\,\|u\|_{s+1,E}^2, \tag{2.3.68}$$

where, $\mathcal{R}_E^{opt} := K_E\Big(1+\Upsilon_E^{(t)} + Pe_E + \min\Big\{\max\Big\{Pe_E;\, p^2\mu_{inv}^2; \Upsilon_E^{(t)}; \Upsilon_E^{(r)}\Big\}; \frac{K_E}{K_E^{\vee}}\,Pe_E^2\Big\}\Big)$.

For simplicity we assume the diffusion coefficient $K(x) \equiv K$. We now discuss the optimality of (2.3.68) in the cases of convection dominated or reaction dominated phenomenon.

(a) In the convection dominated case, ie. $Pe \geq \max\{\Upsilon_E^{(r)}, \Upsilon_E^{(t)}\} \geq p^2\mu_{\text{inv}}^2$, we get,

$$\sum_{E\in\mathcal{T}_h} \frac{h_E}{p\mathbf{b}_E} \|\mathbf{b}\cdot\nabla(u-u_h)\|_{0,E}^2 \;\leq\; C \sum_{E\in\mathcal{T}_h} \big(\frac{h_E}{p}\big)^{2s+1} \|u\|_{s+1,E}^2. \tag{2.3.69}$$

(b) In the reaction dominated case, ie. $\min\{\Upsilon_E^{(r)}, \Upsilon_E^{(t)}\} \geq Pe \geq p^2\mu_{\text{inv}}^2$, we get,

$$\sum_{E\in\mathcal{T}_h} (\sigma+g_0)\, \|u-u_h\|_{0,E}^2 \;\leq\; C \sum_{E\in\mathcal{T}_h} \big(\frac{h_E}{p}\big)^{2s+2} \|u\|_{s+1,E}^2. \tag{2.3.70}$$

Thus we have obtained the optimal order of convergence in both L^2 and H^1 norm respectively. In fact, they are also optimal in the $|||\cdot|||$ norm.

2.4 Numerical experiments

In this section we consider two numerical examples to validate the rate of convergence obtained theoretically from the error estimates (see section 2.3). The nonlinear system of equations obtained from the VEM-SUPG discretization is solved with the help of Newton-GMRES method [54]. We choose constant zero function as our initial guess and the stopping criteria for the Newton's loop is set as 10^{-10}. For both the problems we consider the domain to be $[0,1]\times[0,1]$.

The convergence of VEM-SUPG technique is evaluated in the $L^2(\Omega)$ norm, $H^1(\Omega)$ norm and energy norm denoted by $e_{h,0}$, $e_{h,1}$ and $e_{h,3}$ described as follows,

$$\begin{aligned} e_{h,0}^2 &= \sum_{E\in\mathcal{T}_h} \|u-\Pi_p^0 u_h\|_E^2, \quad e_{h,1}^2 = \sum_{E\in\mathcal{T}_h} \|\nabla(u-\Pi_p^\nabla u_h)\|_E^2, \\ e_{h,3}^2 &= \sum_{E\in\mathcal{T}_h} \Big(\|\sqrt{K}\nabla(u-\Pi_p^\nabla u_h)\|_E^2 + (\sigma+g_0)\|u-\Pi_p^0 u_h\|_E^2 + \tau_E\|\mathbf{b}\cdot\nabla(u-\Pi_p^\nabla u_h)\|_E^2 \Big). \end{aligned}$$

2.4.1 Example 1

Let $\sigma = 2$, $\mathbf{b}(x) = (2,-1)$, and $g(u) = -\dfrac{u}{1+u}$. The source term f is determined by considering the smooth solution $u = xy\sin\pi x\sin\pi y$. We consider the Dirichlet boundary condition specified by the solution u. For this problem we perform our computations over the hexagonal mesh shown in Figure 2.1. The comparison between the unstabilized solution and the solution computed using VEM-SUPG stabilization for the second order VEM is shown in Figure 2.2.

In order to demonstrate the optimal rate of convergence, the error plots computed for the following values of $K = 10^{-3}, 10^{-6}$ and 10^{-9} and VEM order $p = 1, 2, 3$ are shown in Figures 2.3. The results agree well with the theoretical results proved in Section 2.3. In Table 2.1, we present the condition numbers of the Jacobian matrix arising from Newton's method for $K = 10^{-6}$. We can observe that the condition numbers obtained for different mesh sizes h and VEM order $p = 1, 2, 3$ are bounded in the range 10^2 to 10^7. The same behaviour is observed for other values of K which are not shown here for the sake of brevity. We confirm that the obtained solutions are stable and accurate.

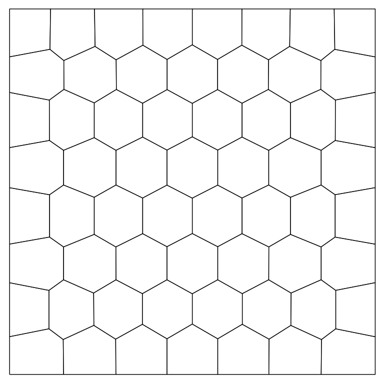
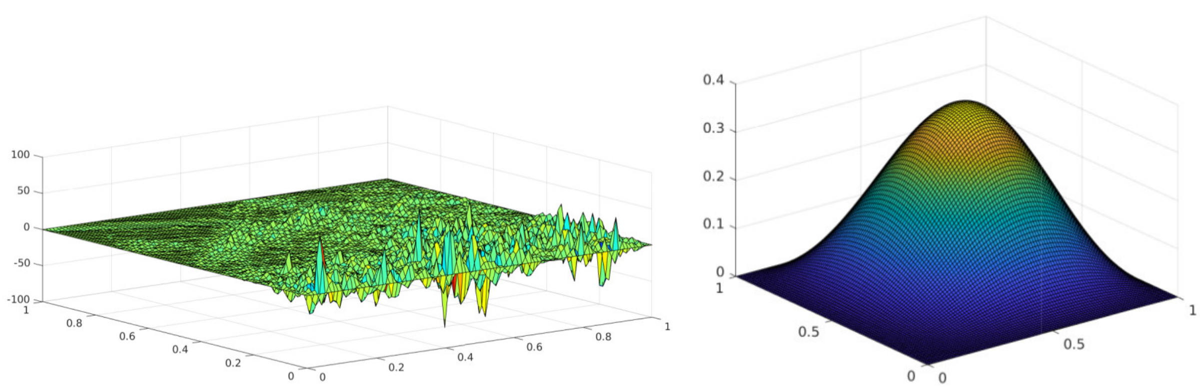


Figure 2.1: Hexagonal mesh for $h = 1/20$.

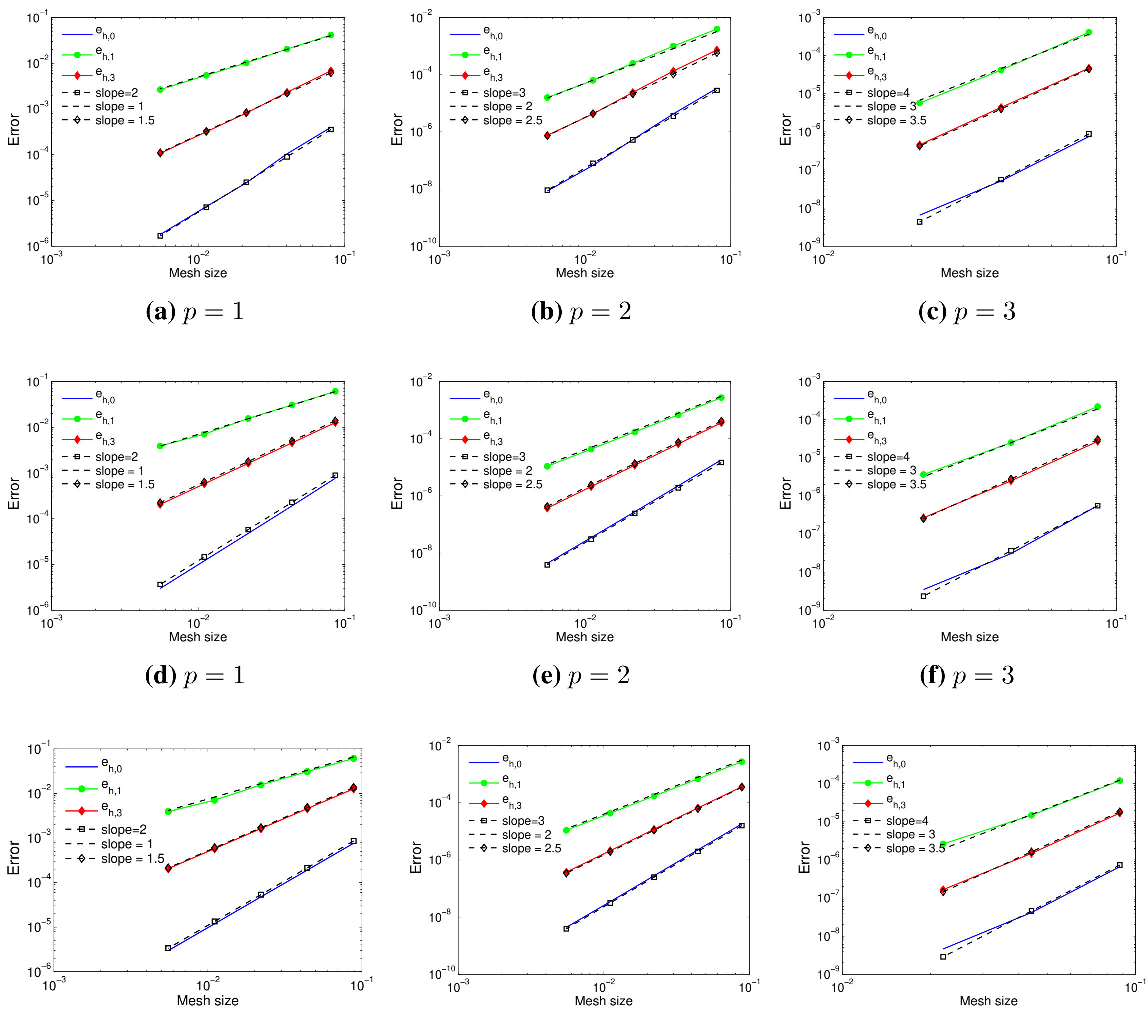


(a) (b)

Figure 2.2: Approximation for $K = 10^{-9}, h = 1/20$ and $p = 2$. On the left, Unstabilized solution, on the right, Stabilized solution.

h	$p=1$	$p=2$	$p=3$
1/12	$4.5e2$	$2.2e3$	$4.5e5$
1/24	$1.5e3$	$6.4e3$	$9.1e5$
1/48	$5.7e3$	$2.2e4$	$1.9e6$
1/96	$2.3e4$	$7.6e4$	$3.7e6$
1/192	$9.1e4$	$2.9e5$	$7.3e6$

Table 2.1: Condition number of Jacobian matrix for $K = 10^{-6}$.



(a) $p = 1$ **(b)** $p = 2$ **(c)** $p = 3$

(d) $p = 1$ **(e)** $p = 2$ **(f)** $p = 3$

(g) $p = 1$ **(h)** $p = 2$ **(i)** $p = 3$

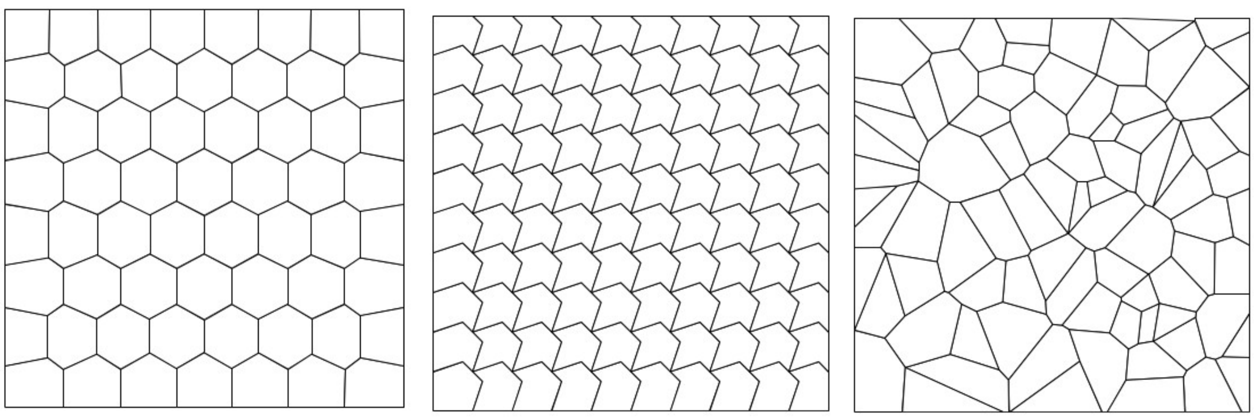
Figure 2.3: Convergence plots with respect to hexagonl mesh for $K = 10^{-3}$ (top), $K = 10^{-6}$ (middle) and $K = 10^{-9}$ (bottom).

2.4.2 Example 2

Let $\sigma = 12$, $\mathbf{b}(x) = (2, 3)$, and $g(u) = u^3$. The source term f is chosen in accordance with the exact solution

$$
\begin{aligned}
u(x,y) =& 16\,x(1-x)y(1-y)\times \\
& \Big[0.5 + \pi^{-1}\text{arctan}\left(200\left(0.25^2 - (x-0.5)^2 - (y-0.5)^2\right)\right)\Big].
\end{aligned} \tag{2.4.1}
$$

For the boundary condition we consider the Dirichlet boundary values prescribed by the exact solution. Since the solution possesses circular internal layer we would like to check the rate of convergence on more general polygonal meshes shown in Figure 2.4 along with its mesh parameters in Table 2.2.



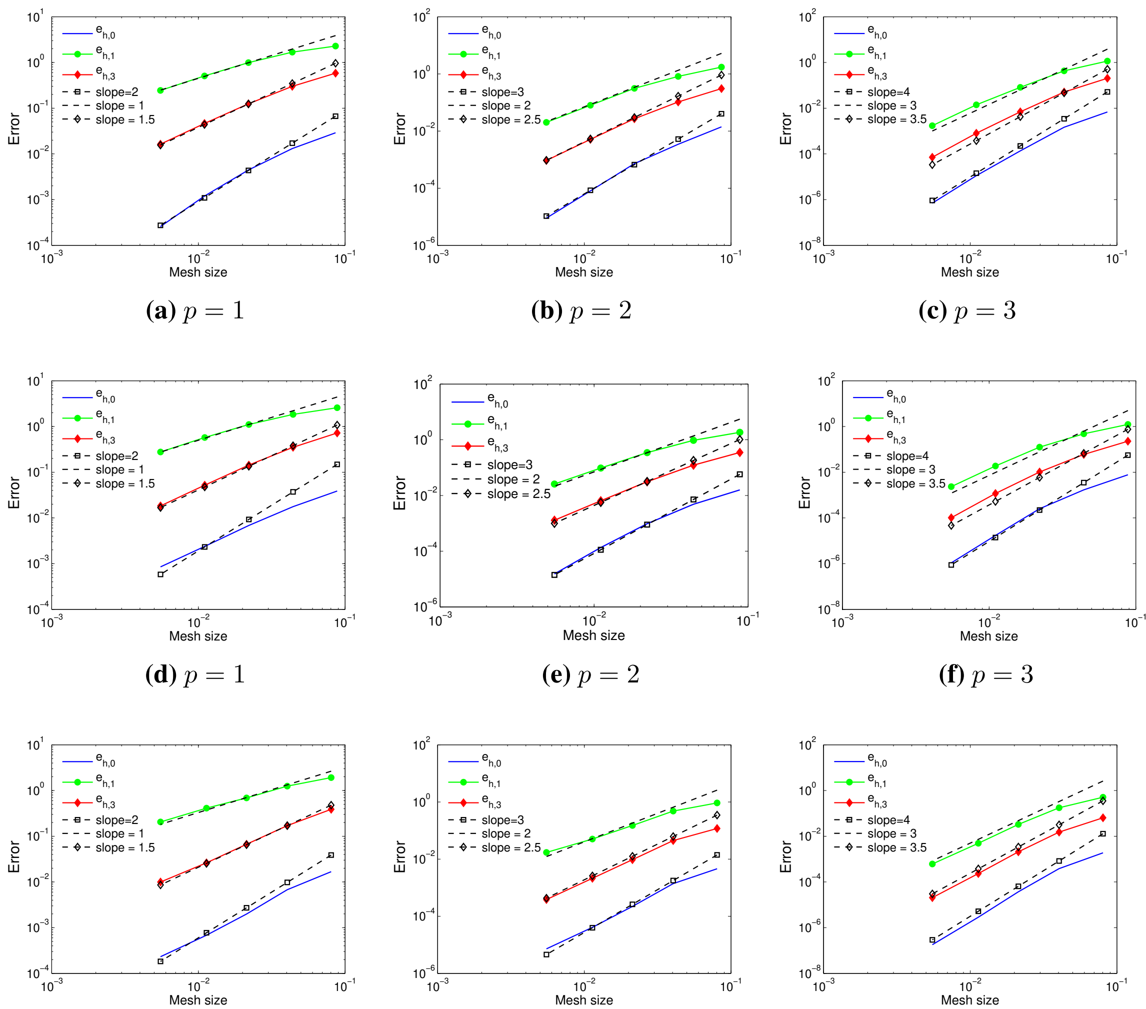
(a) hexagon **(b)** nonconvex **(c)** random voronoi

Figure 2.4: Sample polygonal meshes for $h = 1/5$.

hexagon			nonconvex			random voronoi		
h	dof	N_E	h	dof	N_E	h	dof	N_E
0.0436	3323	1660	0.0442	4801	1600	0.0405	10655	5700
0.0219	13042	6520	0.0221	19201	6400	0.0213	42709	23000
0.0115	51682	25840	0.0110	76801	25600	0.0113	150617	81800
0.0055	205762	102880	0.0054	307201	102400	0.0054	632372	340000

Table 2.2: Mesh parameters with degrees of freedom (dof) and number of elements (N_E).

We have considered VEM of order $p = 1, 2, 3$ for our computations. The convergence plots are shown for $K = 10^{-6}$ in Figures 2.5 and for $K = 10^{-9}$ in Figures 2.6 respectively. From the results we observe the expected rates of convergence. We have computed the condition number of the Jacobian matrix for hexagon, nonconvex and random voronoi meshes by varying mesh size h. For $K = 10^{-6}$ and VEM order $p = 1, 2, 3$ the condition numbers are provided in Table 2.3. Similar to Example 1, we observe that the condition numbers are bounded in the range 10^2 to 10^8. This implies that the obtained solutions are stable and accurate.



(a) $p = 1$ **(b)** $p = 2$ **(c)** $p = 3$

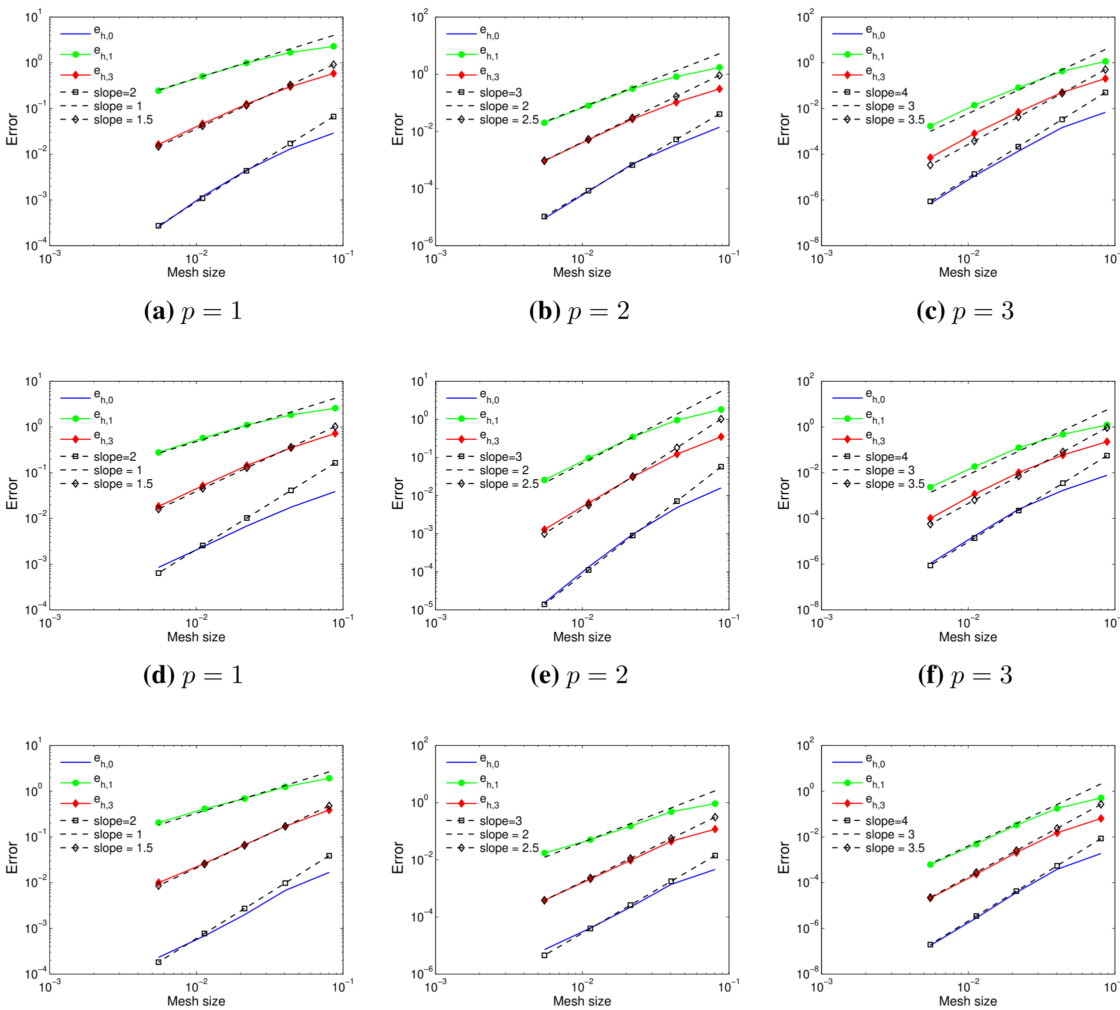
(d) $p = 1$ **(e)** $p = 2$ **(f)** $p = 3$

(g) $p = 1$ **(h)** $p = 2$ **(i)** $p = 3$

Figure 2.5: Convergence plots for hexagonal mesh (top), nonconvex mesh (middle) and random voronoi mesh (bottom) for $K = 10^{-6}$.

In order to make a comparison with Newton-GMRES (NG) method, we have also solved the nonlinear system of equations using first order VEM over random voronoi mesh by fixed point (FP) iteration method. Similar to NG method initial guess is taken as zero function and stopping criteria is considered as 10^{-10}. The results are displayed in Table

2.4 in terms of number of iterations and CPU time in seconds. Even though the number of iterations taken by FP is more than NG, the CPU time taken is relatively less. Similar results are also observed for regular hexagon and nonconvex meshes which are not shown here for the sake of brevity.



(a) $p = 1$ **(b)** $p = 2$ **(c)** $p = 3$

(d) $p = 1$ **(e)** $p = 2$ **(f)** $p = 3$

(g) $p = 1$ **(h)** $p = 2$ **(i)** $p = 3$

Figure 2.6: Convergence plots for hexagonal mesh (top), nonconvex mesh (middle) and random voronoi mesh (bottom) for $K = 10^{-9}$.

2.5 Summary

This chapter has formulated and analysed the virtual element discretisation of the nonlinear convection-diffusion-reaction equation with Streamline upwind Petrov Galerkin sta-

	hexagon			nonconvex			random voronoi		
h	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$
1/12	$1.1e2$	$7.9e2$	$1.6e5$	$1.6e2$	$1.6e3$	$2.7e5$	$4.6e2$	$6.2e3$	$9.1e5$
1/24	$3.7e2$	$1.6e3$	$2.9e5$	$5.4e2$	$2.9e3$	$4.4e5$	$3.1e3$	$1.6e4$	$4.1e6$
1/48	$1.4e3$	$5.2e3$	$5.4e5$	$2.1e3$	$6.7e3$	$7.9e5$	$9.6e3$	$4.4e4$	$7.5e6$
1/96	$5.6e3$	$1.9e4$	$1.1e6$	$7.9e3$	$2.5e4$	$1.4e6$	$4.8e4$	$1.4e5$	$2.9e7$
1/192	$2.3e4$	$7.1e4$	$8.1e6$	$3.1e4$	$9.2e4$	$9.1e6$	$1.8e5$	$5.2e5$	$7.2e7$

Table 2.3: Condition number of Jacobian matrix for $K = 10^{-6}$.

h	NG		FP	
	Iteration	Time	Iterations	Time
$1/12$	5	0.65s	10	0.58s
$1/24$	5	2.52s	10	2.45s
$1/48$	5	13.88s	11	9.86s
$1/96$	5	157.60s	11	40.06s

Table 2.4: Comparison table for NG and FP

bilization. We have suitably added the VEM stabilizer for the nonlinear term to ensure the well-posedness. We deduced an inverse inequality result satisfied by functions in virtual element space with explicit coefficients of h and p. We obtained a virtual element interpolation operator with optimal approximation property in L^2 and H^1 norm for mesh size h and polynomial order p. Error estimate showing optimal rate of convergence in parameters h and p were derived with respect to the natural norm, $|||\cdot|||$. We conducted numerical experiments on two problems. The first problem contains a smooth solution, and the second example consists of a solution possessing circular internal layers. However, we observe a stable solution for both samples, even for a very small diffusion coefficient K. As proved in the theoretical estimates, we attained the optimal convergence rate for higher-order virtual element schemes and various convex and non-convex polygonal meshes.

Chapter 3

A shock-capturing Virtual Element Method for the semilinear convection-diffusion-reaction equation

The SUPG stabilization of VEM discussed in Chapter 2 reduces spurious oscillations along the streamline or flow direction. In literature, we note that sometimes even for linear problems with discontinuities, mere SUPG stabilization of numerical methods do not entirely remove the unphysical oscillations occurring at the layers ([55],[56]). The fluctuations in the SUPG solution is due to the presence of sharp layers not aligned with the flow direction. Thus we must add additional stabilization to the SUPG method to obtain a more accurate approximate solution. In particular, the added stabilization term must efficiently act along the crosswind direction to capture the oscillations. The remedy to this situation is usually called the shock-capturing technique, which adds artificial diffusion in the transverse direction around regions of layers. As we shall see, the shock-capturing term is nonlinear since it involves the residual of the numerical method, which depends on the approximate solution.

The shock capturing stabilization for a one-dimensional singularly perturbed problem and its generalisation to multidimensional model problems is studied in [57]. The exposition and application of shock-capturing technique in the finite element context, for linear advection-diffusion model problems can be found for example in [58], [59], [60], [61] ; for nonlinear convection-diffusion-rection equations in [62], [47] and for hyperbolic system of conservation laws in [63]. A review of the shock-capturing technique, along with the comparison of various choices of proposed shock-capturing parameters, is given in [64].

This chapter studies the shock-capturing stabilization of VEM for the convection-diffusion problems.We first discuss the shock-capturing technique for the linear convection-diffusion-reaction equation. The parameter in the shock-capturing term is a function of scaled residual. To evaluate the norm of the residual we need to know explicit definition of the numerical solution in the virtual element space. But, in VEM we only have knowledge about the polynomial subspace and the functions are identified only through degrees of freedom. Therefore, to estimate the norm of the residual, we introduce suitable polynomial projection operators in the expression of residual, in the VEM setting. In section 3.1, a computable virtual element formulation stabilized with SUPG and shock-capturing term is proposed. From chapter 2, it is evident that a VEM stabilization of the SUPG stabilization is required to ensure the coercivity of the VEM-SUPG formulation. Along similar lines, for the shock-capturing term, we add a VEM stabilizer with appropriate nonlinear coefficients, to ensure stability of the discrete scheme. Surprisingly, shock-capturing approximation of a linear problem produces a nonlinear discrete scheme. We examine the well-posedness of the nonlinear discrete scheme and investigate the efficiency of the shock-capturing method with numerical examples. With this inducement, it would be interesting to analyse the shock-capturing technique for semilinear transport problems which models many scientific/engineering applications.

Subsequently, we describe the shock-capturing stabilization for semilinear convection-diffusion-reaction equation in the VEM framework in section 3.6. Different from the linear case, for approximating the nonlinear reaction functions, an extra VEM stabilizer with suitable linear coefficient is introduced. It would be interesting to verify if the shock-capturing stabilization and the additional VEM stabilsers in the VEM formulation, do not deteriorate the convergence rate. We give a detailed analysis of the nonlinear scheme showing the convergence of the family of discrete solutions to the exact solution and obtaining optimal order error estimate with respect to suitable natural norm.

3.1 Linear model Problem

Let us consider the linear convection-diffusion equation with homogeneous Dirichlet boundary condition:

$$\begin{aligned} -\nabla \cdot (K \nabla u) + \mathbf{b} \cdot \nabla u + \alpha u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.1.1}$$

Here $u(x)$ denotes the unknown where $x \in \Omega \subset \mathbb{R}^2$, $K > 0$, $\mathbf{b} \in W^{1,\infty}(\Omega)^2$ is the velocity field, $\alpha \geq 0$ and $f \in L^2(\Omega)$. We also assume $(\nabla \cdot \mathbf{b})(x) = 0$ a.e in Ω and $K \geq K_0 > 0$.

The bilinear form of equation (3.1.1) is defined as $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ such that

$$B(w,v) = (K\nabla w, \nabla v)_\Omega + (\mathbf{b}\cdot\nabla w, v)_\Omega + (\alpha\, w, v)_\Omega \qquad \forall\, w, v \in H_0^1(\Omega).$$

Using integration by parts on the convective term $(\mathbf{b}\cdot\nabla w, v)$ and the condition $\nabla\cdot\mathbf{b} = 0$, the bilinear form B, can be equivalently redefined as, $\forall\, w, v \in H_0^1(\Omega)$

$$B(w,v) = (K\nabla w, \nabla v)_\Omega + \frac{1}{2}[(\mathbf{b}\cdot\nabla w, v)_\Omega - (w, \mathbf{b}\cdot\nabla v)_\Omega] + (\alpha w, v)_\Omega. \tag{3.1.2}$$

The weak formulation of (3.1.1) is : Find $u \in H_0^1(\Omega)$ such that

$$B(u,v) = (f,v)_\Omega \quad \forall\, v \in H_0^1(\Omega). \tag{3.1.3}$$

The existence and uniqueness of the solution of weak formulation (3.1.3) follows from the Lax-Milgram lemma [65].

3.1.1 VEM Spaces

Consider $\{\mathcal{T}_h\}_{h>0}$ to be a family of polygonal partitioning of Ω satisfying the assumption 1.1 stated in Chapter 1. In our analysis, we use the polynomial projection operators Π_p^∇, Π_p^0 and $\mathbf{\Pi}_{p-1}^0$ defined in (1.3.1), (1.3.2) and (1.3.3), respectively. For approximation we consider the global virtual element space V_h^p given in (1.3.5).

3.2 VEM-SUPG formulation

We tackle the singularly perturbed case i.e., $K \ll 1$, as done in Chapter 2 and proceed to define the SUPG stabilized virtual element discretisation of the formulation (3.1.3) as follows. Find $u_h \in V_h^p$ such that

$$B_{vs}(u_h, v_h) = F_{vs}(v_h) \quad \forall\, v_h \in V_h^p, \tag{3.2.1}$$

where the bilinear form $B_{vs} : V_h^p \times V_h^p \to \mathbb{R}$ is such that,

$$B_{vs}(w_h, v_h) := a_h(w_h, v_h) + b_h(w_h, v_h) + c_h(w_h, v_h) + d_h(w_h, v_h), \tag{3.2.2}$$

with the terms $a_h(\cdot,\cdot)$, $b_h(\cdot,\cdot)$, $c_h(\cdot,\cdot)$, and $d_h(\cdot,\cdot)$ defined as below,

$$
\begin{aligned}
a_h(w_h, v_h) \;&:=\; \sum_{E\in\mathcal{T}_h} \Big[\big(K\boldsymbol{\Pi}^0_{p-1}\nabla w_h, \boldsymbol{\Pi}^0_{p-1}\nabla v_h\big)_E + \tau_E(\mathbf{b}\cdot\boldsymbol{\Pi}^0_{p-1}\nabla w_h, \mathbf{b}\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h)_E \\
&\qquad + (K + \tau_E \mathbf{b}_E^2) S_1^E\left((I-\Pi_k^\nabla)w_h, (I-\Pi_k^\nabla)v_h\right)\Big]. \qquad (3.2.3)\\
b_h(w_h, v_h) \;&:=\; \sum_{E\in\mathcal{T}_h} [(\alpha\Pi_p^0 w_h, \Pi_p^0 v_h)_E + \alpha S_2^E((I-\Pi_k^0)w_h, (I-\Pi_k^0)v_h)]. \qquad (3.2.4)\\
c_h(w_h, v_h) \;&:=\; \frac{1}{2}\sum_{E\in\mathcal{T}_h} \big[(\mathbf{b}\cdot\boldsymbol{\Pi}^0_{p-1}\nabla w_h, \Pi_p^0 v_h)_E - (\Pi_p^0 w_h, \mathbf{b}\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h)_E\big]. \qquad (3.2.5)\\
d_h(w_h, v_h) \;&:=\; \sum_{E\in\mathcal{T}_h} \tau_E\left(-\nabla\cdot K\boldsymbol{\Pi}^0_{p-1}\nabla w_h + \alpha\Pi_p^0 w_h, \mathbf{b}\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h\right)_E. \qquad (3.2.6)
\end{aligned}
$$

and the linear form $F_{vs} : V_h^p \to \mathbb{R}$ is defined as

$$
F_{vs}(v_h) := \sum_{E\in\mathcal{T}_h} \big[(f, \Pi_k^0 v_h)_E + \tau_E(f, \mathbf{b}\cdot\Pi^0_{k-1}\nabla v_h)_E\big]. \qquad (3.2.7)
$$

where $\mathbf{b}_E = \sup_{x\in E}\|\mathbf{b}(x)\|_{\mathbb{R}^2}$, τ_E is the stabilization parameter that is chosen accordingly and, as usual the VEM Stabilizers $S_1^E(\cdot,\cdot)$ and $S_2^E(\cdot,\cdot)$ denotes the symmetric positive bilinear forms defined on $V_E^k \times V_E^k$ by the following,

$$
S_1^E(u_h, v_h) = \sum_{i=1}^{\mathrm{N}} \mathrm{dof}_i(u_h)\,\mathrm{dof}_i(v_h) \quad \text{and} \quad S_2^E(u_h, v_h) = h_E^2\sum_{i=1}^{\mathrm{N}} \mathrm{dof}_i(u_h)\,\mathrm{dof}_i(v_h),
$$

where $\mathrm{dof}_i(u_h)$ denotes the ith degree of freedom of u_h with N denoting the total degrees of freedom. Let there exists non-zero positive constants β_*, β^*, η_* and η^* independent of h_E, such that,

$$
\begin{aligned}
\beta_*(\nabla u_h, \nabla u_h)_E \;&\le\; S_1^E(u_h, u_h) \;\le\; \beta^*(\nabla u_h, \nabla u_h)_E \quad \forall u_h \in \ker(\Pi_p^\nabla), \qquad (3.2.8)\\
\eta_*(u_h, u_h)_E \;&\le\; S_2^E(u_h, u_h) \;\le\; \eta^*(u_h, u_h)_E \qquad \forall u_h \in \ker(\Pi_p^0). \qquad (3.2.9)
\end{aligned}
$$

We introduce the norm $|||\cdot|||$ to be used in our error analysis,

$$
|||v|||^2 := \sum_{E\in\mathcal{T}_h} \left(K\|\nabla v\|_E^2 + \|\sqrt{\alpha}\,v\|_E^2 + \tau_E\|\mathbf{b}\cdot\nabla v\|_E^2\right). \qquad (3.2.10)
$$

We state the local inverse inequality to be used later; there exists a constant C_l such that,

$$
\|\nabla\cdot K\nabla v_h\|_{0,E} \leqslant C_l\, h_E^{-1}\|K\,\nabla v_h\|_{0,E} \qquad \forall\, v_h \in V_h \text{ and } E \in \mathcal{T}_h. \qquad (3.2.11)
$$

For analysis, we assume the following constraints on the local SUPG parameter τ_E :

(G1) $\exists\ \rho \in (0,3)$ independent of $E \in \mathcal{T}_h$ such that

$$(i)\ K\tau_E\, C_l^2 \leqslant \frac{1}{2}\rho\, h_E^2 \quad \text{and} \quad (ii)\ \tau_E\, \alpha \leqslant \frac{1}{2}\,\rho \quad \text{a.e. in}\Omega,$$

where C_l is the same constant used in (3.2.11). An optimal choice for τ_E will be discussed in error analysis for nonlinear model problem in theorem 3.6.

3.3 Well-posedness of VEM-SUPG formulation

In this section we will show the well-posedness of the VEM-SUPG formulation (3.2.1) by first showing the coercivity and then the continuity of bilinear form B_{vs}.

Lemma 3.1. *(Coercivity) The bilinear form $B_{vs}(\cdot,\cdot)$ satisfies the following estimate,*

$$B_{vs}(v_h, v_h) \geqslant C_\rho\, |||v_h|||^2 \quad \forall\, v_h \in V_h, \tag{3.3.1}$$

with $C_\rho = \min\Big\{\beta_*,\ \eta_*,\ \big(1 - \dfrac{\sqrt{\rho}}{2}\big)\Big\} > 0.$

Proof. We estimate the terms of $B_{vs}(\cdot,\cdot)$ one by one. We have,

$$\begin{aligned} a_h(v_h, v_h) &= \sum_{E\in\mathcal{T}_h} \Big[(K\boldsymbol{\Pi}^0_{p-1}\nabla v_h, \boldsymbol{\Pi}^0_{p-1}\nabla v_h)_E + \tau_E(\mathbf{b}\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h, \mathbf{b}\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h)_E \\ &\qquad + (K+\tau_E\mathbf{b}_E^2) S_1^E\left((I-\Pi_k^\nabla)v_h, (I-\Pi_k^\nabla)v_h\right)\Big] \\ &\geq \sum_{E\in\mathcal{T}_h} \left[K\|\Pi^0_{k-1}\nabla v_h\|_E^2 + \tau_E\|\mathbf{b}\cdot\Pi^0_{k-1}\nabla v_h\|_E^2 + \beta_*(K+\tau_E\mathbf{b}_E^2)\|(I-\Pi_k^\nabla)\nabla v_h\|_E^2\right] \\ &\geq \sum_{E\in\mathcal{T}_h} \left[K\|\Pi^0_{k-1}\nabla v_h\|_E^2 + \tau_E\|\mathbf{b}\cdot\Pi^0_{k-1}\nabla v_h\|_E^2 + \beta_*(K+\tau_E\mathbf{b}_E^2)\|(I-\Pi^0_{k-1})\nabla v_h\|_E^2\right]. \end{aligned} \tag{3.3.2}$$

Similarly,

$$b_h(u_h, v_h) \geq \sum_{E\in\mathcal{T}_h} \alpha\left(\|\Pi_p^0 v_h\|_E^2 + \eta_*\|(I-\Pi_k^0)v_h\|_E^2\right). \tag{3.3.3}$$

Note that $c_h(v_h, v_h) = 0$. Next, estimating the last term of B_{vs}, we have for some $\lambda > 0$,

$$\begin{aligned} |d_h(v_h, v_h)| &\leqslant \sum_{E\in\mathcal{T}_h} \left|\tau_E\left(-\nabla\cdot K\boldsymbol{\Pi}^0_{p-1}\nabla v_h + \alpha\,\Pi_p^0 v_h, \mathbf{b}\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h\right)_E\right| \\ &\leqslant \sum_{E\in\mathcal{T}_h} \tau_E\,\| -\nabla\cdot K\boldsymbol{\Pi}^0_{p-1}\nabla v_h + \alpha\Pi_p^0 v_h\|_{0,E}\,\|\mathbf{b}\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h\|_{0,E} \\ &\leqslant \sum_{E\in\mathcal{T}_h} \left(\frac{\tau_E}{2\lambda}\| -\nabla\cdot K\boldsymbol{\Pi}^0_{p-1}\nabla v_h\|_E^2 + \frac{\tau_E}{2\lambda}\|\alpha\Pi_p^0 v_h\|_E^2 + \frac{\lambda\,\tau_E}{2}\|\mathbf{b}\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h\|_E^2\right). \end{aligned}$$

Applying the inverse inequality (3.2.11), (G1) and then choosing $\lambda = \sqrt{\rho}$, we get,

$$
\begin{aligned}
|d_h(v_h, v_h)| &\leqslant \sum_{E\in\mathcal{T}_h} \Big(\frac{\rho}{2\lambda} K \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|^2_E + \frac{\rho}{2\lambda}\|\sqrt{\alpha}\,\Pi^0_p v_h\|^2_E + \frac{\lambda\,\tau_E}{2}\|\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\|^2_E \Big) \\
&\leqslant \sum_{E\in\mathcal{T}_h} \left(\frac{\sqrt{\rho}}{2} K \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|^2_E + \frac{\sqrt{\rho}}{2}\|\sqrt{\alpha}\,\Pi^0_p v_h\|^2_E + \frac{\sqrt{\rho}\,\tau_E}{2}\,\|\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\|^2_E \right).
\end{aligned} \tag{3.3.4}
$$

Combining the estimates (3.3.2), (3.3.3) and (3.3.4), we have,

$$
\begin{aligned}
B_{vs}(v_h, v_h) \geqslant & \left(1 - \frac{\sqrt{\rho}}{2}\right) \sum_{E\in\mathcal{T}_h} \big[(K\|\Pi^0_{k-1}\nabla v_h\|^2_E + \tau_E\|\mathbf{b}\cdot\Pi^0_{k-1}\nabla v_h\|^2_E + \alpha\|\Pi^0_p v_h\|^2_E \,\big] + \\
& \sum_{E\in\mathcal{T}_h} \big[\, \beta_*(K + \tau_E\mathbf{b}^2_E)\|(I - \Pi^0_{k-1})\nabla v_h\|^2_E + \eta_*\,\alpha\|(I-\Pi^0_k)v_h\|^2_E \,\big] \\
\geqslant & \min\Big\{\beta_*,\, \eta_*,\, \big(1 - \frac{\sqrt{\rho}}{2}\big)\Big\} \sum_{E\in\mathcal{T}_h} \Big(K\|\nabla v_h\|^2_E + \tau_E\|\mathbf{b}\cdot\nabla v_h\|^2_E + \alpha\|v_h\|^2_E \Big).
\end{aligned}
$$

Thus, we obtain the estimate (3.3.1), proving the coercivity. □

Lemma 3.2. *For $u \in H^1_0(\Omega)$ with $(\nabla\cdot K\nabla u)_{|E} \in L^2(E)$, $\forall\, E \in \mathcal{T}_h$ and $v_h \in V_h$ we have,*

$$
|B_{vs}(u, v_h)| \leqslant C_{vs}\,\gamma(u)\,|||v_h|||, \tag{3.3.5}
$$

where C_{vs} is a constant depending on $K, \boldsymbol{b}$, and α, but independent of h and τ_E, and

$$
\gamma(u) := |||u||| + \Big(\sum_{E\in\mathcal{T}_h} \min\left\{ \frac{1}{\tau_E}; \frac{\boldsymbol{b}^2_E}{K_0} \right\} \|u\|^2_E \Big)^{\frac{1}{2}}. \tag{3.3.6}
$$

Proof. Using the inequality (3.2.8), we estimate,

$$
\begin{aligned}
|a_h(u, v_h)| \leqslant & \sum_{E\in\mathcal{T}_h} \Big[\big|\,(K\mathbf{\Pi}^0_{p-1}\nabla u, \mathbf{\Pi}^0_{p-1}\nabla v_h)_E\big| + \big|\tau_E(\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla u, \mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h)_E\big| \\
& + \beta^*(K + \tau_E\mathbf{b}^2_E)\|(I - \Pi^\nabla_k)\nabla u\|_{0,E}\,\|(I - \Pi^\nabla_k)\nabla v_h\|_{0,E} \Big] \\
\leqslant & \sum_{E\in\mathcal{T}_h} (1+\beta^*)\,K\|\nabla u\|_{0,E}\|\nabla v_h\|_{0,E} + \left(\frac{\mathbf{b}^2_E\tau_E}{K_0}\right)(1+\beta^*)K\|\nabla u\|_{0,E}\|\nabla v_h\|_{0,E} \\
\leqslant & (1+\beta^*)\sum_{E\in\mathcal{T}_h} K\|\nabla u\|_{0,E}\|\nabla v_h\|_{0,E} + \left(\frac{\mathbf{b}^2_E\,\tau_E\,\alpha}{K_0\,\alpha}\right) K\|\nabla u\|_{0,E}\|\nabla v_h\|_{0,E} \\
\leqslant & (1+\beta^*)\left(1 + \max_{E\in\mathcal{T}_h}\left(\frac{\mathbf{b}^2_E\,\rho}{K_0\,\alpha}\right)\right) \sum_{E\in\mathcal{T}_h} K\|\nabla u\|_{0,E}\|\nabla v_h\|_{0,E} \qquad \text{(use } (ii) \text{ of (G1))}
\end{aligned}
$$

Using Holder's inequality, we get,

$$|a_h(u, v_h)| \leqslant (1+\beta^*)\left(1+\max_{E\in\mathcal{T}_h}\left(\frac{\mathbf{b}_E^2\,\rho}{K_0\,\alpha}\right)\right)|||u|||\;|||v_h|||. \tag{3.3.7}$$

Similarly using (3.2.9), we have,

$$|b_h(u, v_h)| \leqslant (1+\eta^*)\,|||u|||\;|||v_h|||. \tag{3.3.8}$$

Consider the third term $|c_h(u, v_h)|$, we note,

$$|c_h(u, v_h)| \leqslant \frac{1}{2}\sum_{E\in\mathcal{T}_h}|(\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla u, \Pi^0_p v_h)_E)| + \frac{1}{2}\sum_{E\in\mathcal{T}_h}|(\Pi^0_p u, \mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h)_E|. \tag{3.3.9}$$

Estimating the first term of (3.3.9), we get,

$$\begin{aligned}\frac{1}{2}\sum_{E\in\mathcal{T}_h}|(\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla u, \Pi^0_p v_h)_E)| &\leqslant \frac{1}{2}\sum_{E\in\mathcal{T}_h}\left(\frac{\mathbf{b}_E}{\sqrt{K_0\,\alpha}}\right)\sqrt{K}\|\nabla u\|_{0,E}\sqrt{\alpha}\|v_h\|_{0,E}\\ &\leqslant \max_{E\in\mathcal{T}_h}\left(\frac{\mathbf{b}_E}{\sqrt{K_0\alpha}}\right)|||u|||\;|||v_h|||. \end{aligned} \tag{3.3.10}$$

The second term of (3.3.9) is estimated in two different ways namely,

$$\begin{aligned}\frac{1}{2}\sum_{E\in\mathcal{T}_h}|(\Pi^0_p u, \mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h)_E| &\leqslant \frac{1}{2}\sum_{E\in\mathcal{T}_h}\left[\frac{1}{\sqrt{\tau_E}}\|u\|_{0,E}\tfrac{\mathbf{b}_E\sqrt{\tau_E\,\alpha}}{\sqrt{K_0\,\alpha}}\sqrt{K}\|\nabla v_h\|_{0,E}\right]\\ &\leqslant \left(\max_{E\in\mathcal{T}_h}\frac{\mathbf{b}_E\sqrt{\rho}}{\sqrt{K_0\,\alpha}}\right)\left(\sum_{E\in\mathcal{T}_h}\frac{1}{\tau_E}\|u\|^2_E\right)^{\frac{1}{2}}|||v_h|||. \end{aligned} \tag{3.3.11}$$

$$\begin{aligned}\frac{1}{2}\sum_{E\in\mathcal{T}_h}|(\Pi^0_p u, \mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h)_E| &\leqslant \frac{1}{2}\sum_{E\in\mathcal{T}_h}\|u\|_{0,E}\frac{\mathbf{b}_E}{\sqrt{K_0}}\sqrt{K}\|\nabla v_h\|_{0,E}\\ &\leqslant \Big(\sum_{E\in\mathcal{T}_h}\frac{\mathbf{b}_E^2}{K_0}\|u\|^2_E\Big)^{\frac{1}{2}}|||v_h|||. \end{aligned} \tag{3.3.12}$$

Combining the estimates in (3.3.10),(3.3.11) and (3.3.12) we have,

$$\begin{aligned}|c_h(u, v_h)| \;\leqslant\; &\max_{E\in\mathcal{T}_h}\left(\frac{\mathbf{b}_E}{\sqrt{K_0\alpha}}\right)|||u|||\;|||v_h|||\;+\\ &\max\left(1;\max_{E\in\mathcal{T}_h}\frac{\mathbf{b}_E\sqrt{\rho}}{\sqrt{K_0\alpha}}\right)\left(\sum_{E\in\mathcal{T}_h}\min\left\{\tfrac{1}{\tau_E};\frac{\mathbf{b}_E^2}{K_0}\right\}\|u\|^2_E\right)^{\frac{1}{2}}|||v_h|||. \end{aligned} \tag{3.3.13}$$

Now, we estimate,

$$
\begin{aligned}
|d_h(u,v_h)| &\leqslant \sum_{E\in\mathcal{T}_h} (\tau_E\| -\nabla\cdot K\,\mathbf{\Pi}^0_{p-1}\nabla u\|_{0,E} + \tau_E\|\alpha\,\Pi^0_p u\|_{0,E})\|\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{0,E} \\
&\leqslant \sum_{E\in\mathcal{T}_h} (\tau_E\| -\nabla\cdot K\,\mathbf{\Pi}^0_{p-1}\nabla u\|_{0,E}\|\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{0,E}) \\
&\qquad + (\tau_E\|\alpha\,\Pi^0_p u\|_{0,E}\|\mathbf{b}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{0,E})
\end{aligned}
$$

Using the inverse inequality (3.2.11) and the assumptions in (G1), we get,

$$
\begin{aligned}
|d_h(u,v_h)| &\leqslant \sum_{E\in\mathcal{T}_h} \left(\frac{\rho\,\mathbf{b}_E}{2\sqrt{K_0\,\alpha}}\right)\left(K\|\nabla u\|_{0,E}\|\nabla v_h\|_{0,E} + \sqrt{\alpha}\|u\|_{0,E}\sqrt{K}\|\nabla v_h\|_{0,E}\right) \\
&\leqslant \max_{E\in\mathcal{T}_h}\left(\frac{\rho\,\mathbf{b}_E}{\sqrt{K_0\,\alpha}}\right) |||u|||\;|||v_h|||. \qquad (3.3.14)
\end{aligned}
$$

Let $\mathcal{B} := \max_{E\in\mathcal{T}_h}\mathbf{b}_E$ and $\mathcal{C} := (\,K_0\alpha\,)^{-\frac{1}{2}}$. Then, combining the estimates (3.3.7), (3.3.8), (3.3.13) and (3.3.14) we obtain the result (3.3.5) with

$$
C_{vs} = \max\left\{ [(1+\beta^*)(1+\rho\,\mathcal{B}^2\mathcal{C}^2) + (1+\eta^*) + \mathcal{B}\mathcal{C} + \rho\,\mathcal{B}\mathcal{C}]\;;\;\max(1\,,\sqrt{\rho}\,\mathcal{B}\mathcal{C})\right\}.
$$

Hence the lemma is proved. □

3.4 VEM-SUPG with shock-capturing

In this section we formulate the shock-capturing technique for the VEM discretization of our model problem (3.1.1) and discuss the results concerning the proof of existence of discrete solution. Consider the following term,

$$
\begin{aligned}
T_{sc}(w;u,v) \;&=\; \sum_{E\in\mathcal{T}_h}\left(\delta_E(w)\,N_{sc}\,\mathbf{\Pi}^0_{p-1}\nabla u, \mathbf{\Pi}^0_{p-1}\nabla v\right)_E \\
&\qquad + \|N_{sc}\|_\infty\,\delta_E(\Pi^0_0 w)\,S^E_1\left((I-\Pi^\nabla_k)u,(I-\Pi^\nabla_k)v\right). \qquad (3.4.1)
\end{aligned}
$$

where, N_{sc} is a symmetric positive definite matrix function such that $\|(N_{sc})_{(ij)}\|_{\infty,\Omega} \le 1$ and $\delta_E(w)$ is chosen satisfying the following condition:

(G2) We suppose that $\delta_E(w)$ depends continuously on w and

$$
0 \le \delta_E(w) \le M_E(h_E) \quad \text{with} \quad \lim_{h\to 0} M_E(h) = 0. \qquad (3.4.2)
$$

We consider the following shock-capturing formulation, find $u_h \in V_h^p$ such that,

$$B_{vs}(u_h, v_h) + T_{sc}(u_h; u_h, v_h) = F_{vs}(v_h) \qquad \forall\, v_h \in V_h^p, \tag{3.4.3}$$

Let us consider the following two choices for δ_E and N_{sc} concerning addition of the isotropic and anisotropic diffusion.

Case I : Anisotropic diffusion

$$\delta_E(w) = \frac{\sigma_E(w)\|\, L_{sc}(w) - f\|_{0,E}}{\kappa + \|\nabla \Pi_p^{\nabla} w\|_{0,E}}, \quad N_{sc} := \begin{cases} \mathbf{I} - \dfrac{\mathbf{b}\otimes\mathbf{b}}{|\mathbf{b}|}\,, \mathbf{b} \neq \mathbf{0} \\ \mathbf{0}\,, \mathbf{b} = \mathbf{0} \end{cases} \tag{3.4.4}$$

Case II : Isotropic diffusion

$$\delta_E(w) = \frac{\sigma_E(w)\|\, L_{sc}(w) - f\|_E^2}{\left[\kappa + \left(\|\Pi_p^0 w\|_E^2 + \|\nabla \Pi_p^{\nabla} w\|_E^2\right)^{\frac{1}{2}}\right]^2}, \quad N_{sc} := \mathbf{I} \tag{3.4.5}$$

where, $\sigma_E(w) \geq 0$, $\kappa \geq 0$ are chosen such that δ_E satisfies (3.4.2), and

$$L_{sc}(w) := -\nabla \cdot (K\, \mathbf{\Pi}_{p-1}^0 \nabla w) + \mathbf{b} \cdot\, \mathbf{\Pi}_{p-1}^0 \nabla w + \alpha\, \Pi_p^0 w. \tag{3.4.6}$$

We make a particular choice for σ_E as follows (refer [55]),

$$\sigma_E(w) := l_0 h_E \max\left\{0, \beta - \frac{2K}{h_E R_E^*(w)}\right\}, \tag{3.4.7}$$

where, $R_E^*(w) := \dfrac{\|L_{sc}(w) - f\|_{0,E}}{\left[\kappa + \left(\|\Pi_p^0 w\|_E^2 + \|\nabla \Pi_p^{\nabla} w\|_E^2\right)^{\frac{1}{2}}\right]}$ and parameters $\{\, l_0,\, \kappa,\, \beta\,\} \subset (0,1)$.

Remark 3.1. The effect of $\delta_E(w)$ becomes significant only when the residual $\|L_{sc}(w) - f\|_{0,E}$ is very large.

Remark 3.2. The limiter function δ_E depends on u_h. Thus with shock-capturing term in equation (3.4.1), the discrete formulation reduces to nonlinear system of equations. This significantly increases the computational cost of solving a linear model problem.

Now, we proceed to prove the existence of a numerical solution for the scheme (3.4.3) by the following theorem.

Theorem 3.1. *The shock-capturing scheme (3.4.3) has at least one solution* $u_h \in V_h^p$

satisfying the condition,

$$|||u_h|||^2 + T_{sc}(u_h; u_h, u_h) \leqslant C\, |||f|||_*^2 \tag{3.4.8}$$

with the dual norm $|||f|||_* := \sup_{v_h \in V_h^p \setminus \{0\}} \dfrac{F_{vs}(v_h)}{|||v_h|||}$.

Proof. We use a variant of Brouwer's fixed point theorem (see [66], II, Lemma 1.4) to show the existence of a solution.

For this, let us define an inner product on V_h^p as $\langle v_h, v_h \rangle := (\nabla v_h, \nabla v_h)$ and let $P : V_h^p \longrightarrow V_h^p$ be an operator, such that,

$$\langle Pu_h, v_h \rangle = (\nabla Pu_h, \nabla v_h) = B_{vs}(u_h, v_h) + T_{sc}(u_h; u_h, v_h) - F_{vs}(v_h). \tag{3.4.9}$$

Using lemma (3.1) and Young's inequality we get,

$$\begin{aligned} \langle Pv_h, v_h \rangle =& T_{sc}(v_h; v_h, v_h) + B_{vs}(v_h, v_h) - F_{vs}(v_h) \\ \geqslant& T_{sc}(v_h; v_h, v_h) + C_\rho |||v_h|||^2 - |||f|||_* \, |||v_h||| \\ \geqslant& T_{sc}(v_h; v_h, v_h) + \frac{C_\rho}{2} |||v_h|||^2 - \frac{1}{2C_\rho} |||f|||_*^2 \end{aligned} \tag{3.4.10}$$

We conclude that $\langle Pv_h, v_h \rangle > 0$ for all $v_h \in V_h$ with $\langle v_h, v_h \rangle = \|\nabla v_h\|^2 > \tilde{C}\, C_\rho\, |||f|||_*$, for some constant $\tilde{C} > 0$. Clearly, F_{vs} is continuous. Also lemma 3.2 and the assumption (G2) imply the continuity of B_{vs} and T_{sc}. Thus, we get that the operator P is continuous. Then, using a variant of Brouwer's fixed point theorem (see [66]) we get atleast one solution u_h satisfying $P(u_h) = 0$. This inturn implies the existence of a solution of the discrete problem and finally the estimate (3.4.8) is obtained by using $P(u_h) = 0$ in the inequality (3.4.10). □

Remark 3.3. We have shown the existence of atleast one solution for the shock-capturing technique, but unfortunately the uniqueness result is still open. If we assume that δ_E is Lipschitz continuous then using Banach fixed point theorem we can prove the uniqueness. But this condition restricts the choice of δ_E for practical applications. On the otherhand, using the result of Schauder fixed point theorem [67] a corresponding result using Brouwer's fixed point theorem with specific assumptions on δ_E the uniqueness result can be proved. Once again this imposes severe restrictions on δ_E.

Remark 3.4. We would also like to mention that the choice of δ_E given in equation (3.4.4) does not satisfy the Lipschitz continuity.

3.5 Numerical experiments

In this section we illustrate the performance of shock-capturing technique with an example. We would like to make the following choice for the stabilization parameter τ_E proposed in [54],

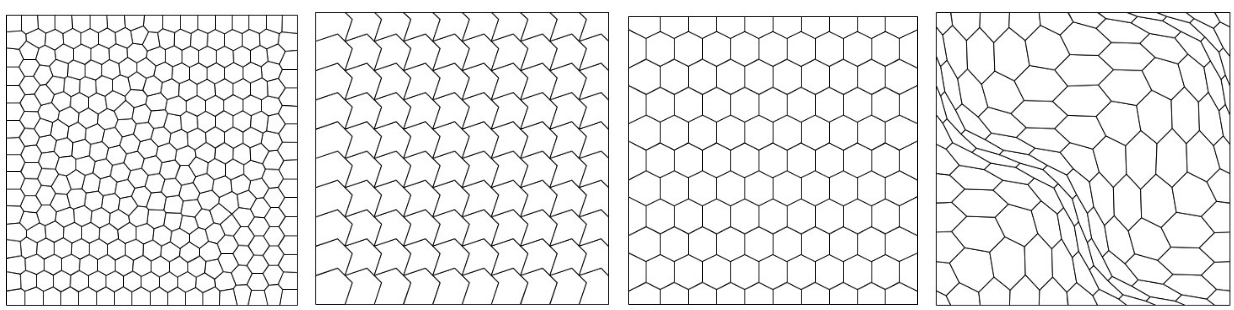
$$\tau_E = \min\left\{\frac{h_E}{|\mathbf{b}|}; \frac{1}{|\alpha|}; \frac{h_E^2}{K}\right\} \tag{3.5.1}$$

The reduced nonlinear algebraic system of equations can be solved by the application of inexact Newton-GMRES algorithm [68]. Since this approach is very expensive we consider solving the scheme (3.4.3) using the following simple iterative technique,

$$n \in \mathbb{N}, \qquad B_{vs}(U^{n+1}, v) + T_{sc}(U^n; U^{n+1}, v) = F_{vs}(v) \qquad \forall\, v \in V_h^p \tag{3.5.2}$$

The well-posedness of this iterative technique is discussed in [55].

For our numerical experiment we consider four different type of meshes namely, smoothed Voronoi, nonconvex polygons, regular hexagons and distorted hexagons respectively shown in figure 3.1. We use VEM of order $k = 1$ and $k = 2$ for our computations.



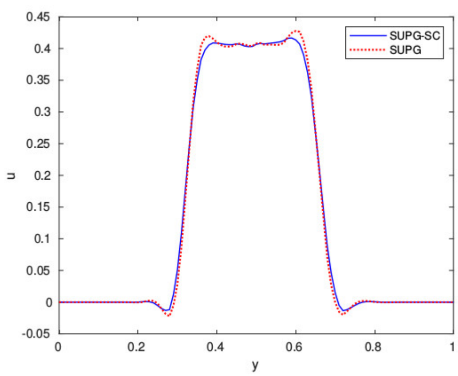
(a) Smoothed Voronoi **(b)** Nonconvex polygons **(c)** Regular hexagons **(d)** Distorted hexagons

Figure 3.1: Polygonal meshes

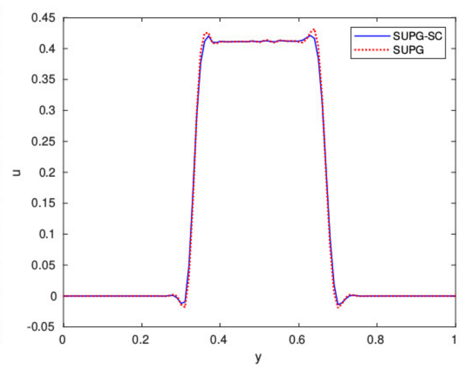
3.5.1 Example 1

We consider a stationary linear convection-diffusion problem. Let $\Omega = (0,1)^2$, $K = 10^{-6}$, $\mathbf{b} = (-y, x)$, $\alpha = 1$, and $f \equiv 0$, in equation (3.1.1). We specify the discontinuous boundary conditions as follows : the Dirichlet condition $u(x,y) = 1$ for $x \in (\frac{1}{3}, \frac{2}{3})$, $y = 0$ and $u(x,y) = 0$ on the remaining parts of lower boundary as well as on the right and upper boundary; assume the homogeneous Neumann condition on the left boundary, ie. $\frac{\partial u(x,y)}{\partial \mathbf{n}} = 0$ for $x = 0$, $y \in (0,1)$, where $\mathbf{n}$ is the unit outerward normal. The discontinuous profile specified on the boundary is carried over to the characteristic curves and the solution develops interior layers.

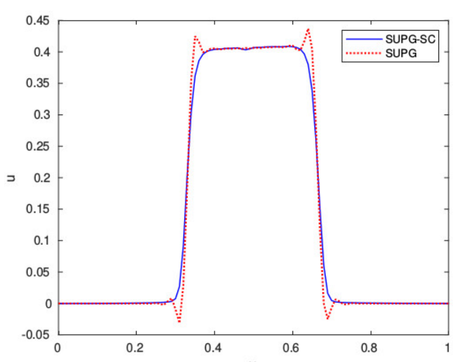
To present our numerical results we denote SUPG with shock-capturing and without shock-capturing as SUPG-SC and SUPG respectively. We choose the following values in the equation (3.4.4) as $l_0 = 0.2$, $\beta = 0.7$ and $\kappa = 10^{-4}$. The iterative scheme (3.5.2) is used for solving the nonlinear system with tolerance 10^{-7}. We note that the solution has two interior layers that are efficiently damped by the VEM-SUPG with shock capturing method on both the orders $k = 1$ and $k = 2$. The cross-section plots of the solution at the left outflow boundary for both SUPG and SUPG-SC are shown in figures 3.2-3.5.



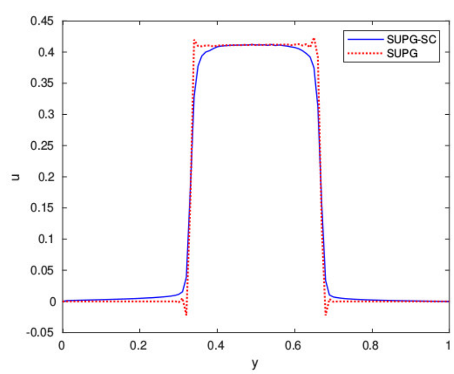
(a) $k = 1,\ h = 1/40$



(b) $k = 1,\ h = 1/80$

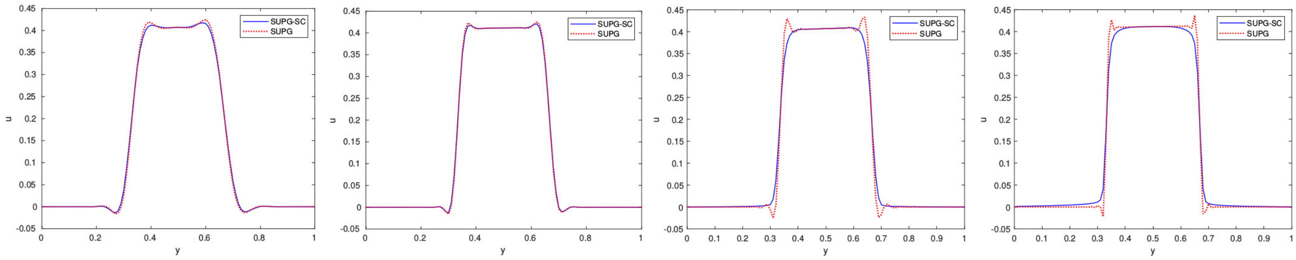


(c) $k = 2,\ h = 1/40$



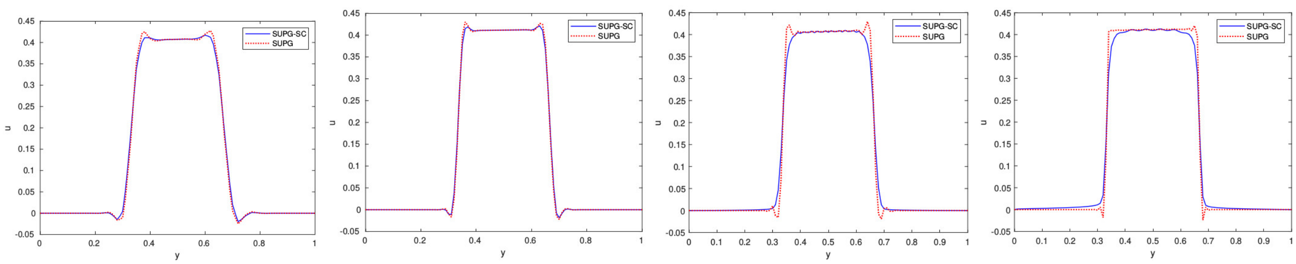
(d) $k = 2,\ h = 1/80$

Figure 3.2: Smoothed Voronoi: The cross-section plots of the solution at the left outflow boundary.



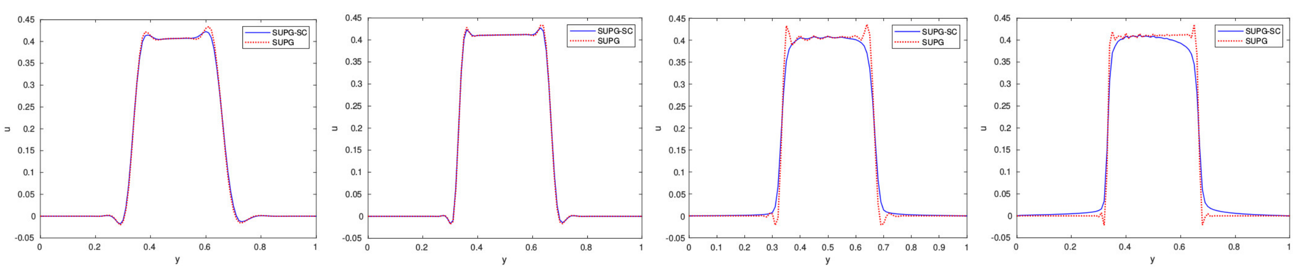
(a) $k = 1,\ h = 1/40$ **(b)** $k = 1,\ h = 1/80$ **(c)** $k = 2,\ h = 1/40$ **(d)** $k = 2,\ h = 1/80$

Figure 3.3: Nonconvex polygons: The cross-section plots of the solution at the left outflow boundary.



(a) $k = 1,\ h = 1/40$ **(b)** $k = 1,\ h = 1/80$ **(c)** $k = 2,\ h = 1/40$ **(d)** $k = 2,\ h = 1/80$

Figure 3.4: Regular hexagons: The cross-section plots of the solution at the left outflow boundary.



(a) $k = 1,\ h = 1/40$ **(b)** $k = 1,\ h = 1/80$ **(c)** $k = 2,\ h = 1/40$ **(d)** $k = 2,\ h = 1/80$

Figure 3.5: Distorted hexagons: The cross-section plots of the solution at the left outflow boundary.

To compare the results with finite element method we consider the mesh of structured triangles shown in figure 3.6. We show the cross-section plots of both FEM and VEM at the left outflow boundary for order $k = 1$ in figure 3.7. We can observe that VEM performs similar to FEM in reducing the oscillations along the sharp layers.

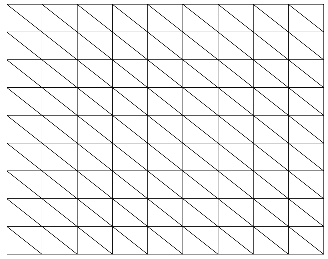
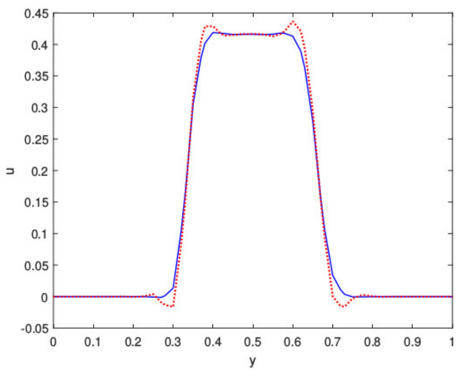
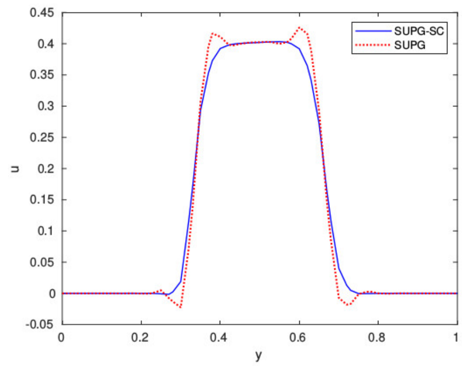


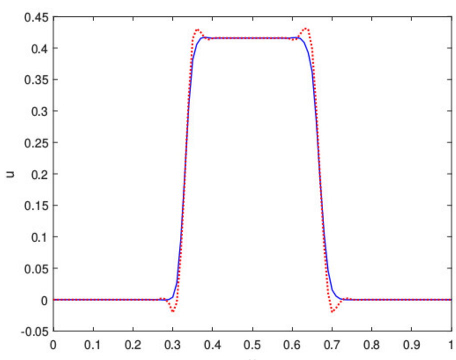
Figure 3.6: Sample structured triangle mesh.



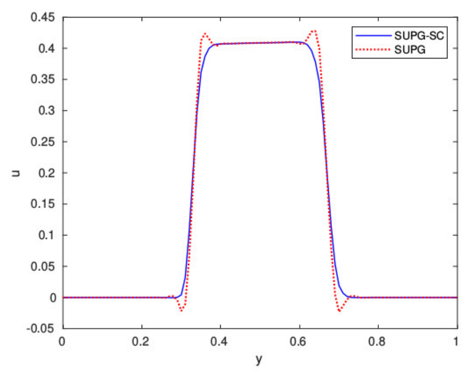
(a) $FEM:\ k=1,\ h=1/40$



(b) $VEM:\ k=1,\ h=1/40$



(c) $FEM: k=1,\ h=1/80$



(d) $VEM: k=1,\ h=1/80$

Figure 3.7: Structured triangle: The cross-section plots of the solution at the left outflow boundary.

3.5.2 Example 2

In this example we consider the problem (see Example 4.1) discussed in [47]. Let $\Omega = (0,1)^2$, $K = 10^{-6}$, $\mathbf{b} = \dfrac{1}{\sqrt{5}}(1,2)^T$ with added nonlinear reaction term u^4. We consider the exact solution as $u(x) = \frac{1}{2}\left(1 - \tanh\left(\frac{2x_1 - x_2 - \frac{1}{4}}{\sqrt{5\,K}}\right)\right)$. This solution exhibits an interior layer with thickness $\mathcal{O}(\sqrt{K}\,|\ln K|)$. We use Dirichlet boundary values prescribed by the solution. In order to make a comparison with finite element method we have considered regular triangular meshes for the numerical computation. We present a result (see table 3.1) depicting the errors evaluated in $|||\cdot|||$ along with roc i.e., the rate of convergence. From this we observe that our proposed method performs better than the method discussed in the paper [47].

Table 3.1: Comparison of errors in $|||\cdot|||$ and the rate of convergence (roc).

Order $k = 2$.				
	SC-CD (*Table 1*,[47])		SUPG-SC (VEM)	
h	$\|\|\|\cdot\|\|\|$	*roc*	$\|\|\|\cdot\|\|\|$	*roc*
$\frac{1}{4}$	1.70e-01	$*$	1.43e-01	$*$
$\frac{1}{8}$	1.32e-01	0.36	1.02e-01	0.48
$\frac{1}{16}$	1.13e-01	0.22	7.14e-02	0.52
$\frac{1}{32}$	9.05e-02	0.32	5.33e-02	0.42
$\frac{1}{64}$	7.05e-02	0.36	3.85e-02	0.47
$\frac{1}{128}$	5.37e-02	0.39	2.67e-02	0.53

So far, we have proposed a computable shock-capturing method stabilized VEM formulation of linear convection-diffusion-reaction equation. The resulting discrete scheme turned out to be nonlinear. Hence to overcome the cost of solving a nonlinear system of algebraic equations, we have used the simple iterative technique. From the numerical examples, we observe that the performance of the shock-capturing technique is consistent for the meshes considered. In particular, the reduction of spurious oscillations by the shock-capturing terms is much more evident for the VEM of order $k = 2$.

As mentioned earlier, a lot of practical applications are studied with the help of nonlinear transport equations. In the remaining sections, we perform theoretical and numerical analysis of the shock-capturing stabilization of VEM for semilinear convection-diffusion-reaction equations.

3.6 Semilinear model Problem

Let us consider the following model equation on a bounded domain $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned} -\nabla \cdot (\mathcal{D}\nabla u) + \overrightarrow{\beta} \cdot \nabla u + \Psi(u) &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.6.1}$$

where u is the solution variable, usually representing concentration of a specific particle in a medium, $\mathcal{D}$ is the diffusion parameter, $\overrightarrow{\beta}$ is the convection/velocity field, $\Psi(\cdot)$ is the reaction term which is a nonlinear function of u and f is the source/sink function of u.

Without loss of generality we let the solution u of (3.6.1) to be non-negative and bounded above i.e., $u_0 \leq u \leq u_1$ with $u_0 \geq 0$. In our analysis we assume that $f \in L^2(\Omega)$, $\mathcal{D} \in L^\infty(\Omega)$ and $\overrightarrow{\beta} \in [W^{1,\infty}(\Omega)]^2$, with $\mathcal{D}(x) \geq \mathcal{D}_0 > 0$, $(\nabla \cdot \overrightarrow{\beta})(x) = 0$ for a.e $x \in \Omega$. On the nonlinear function Ψ we suppose that

$$\Psi \in C^1(\mathbb{R}) \;\; \text{with} \quad \Psi(0) = 0, \quad \Psi'(s) \geq \Psi_0 > 0 \qquad \text{for} \;\; s \in \mathbb{R}^+. \tag{3.6.2}$$

The variational formulation of (3.6.1) is given by : Find $u \in H_0^1(\Omega)$ such that

$$(\mathcal{D}\,\nabla u, \nabla v)_\Omega + (\overrightarrow{\beta} \cdot \nabla u, v)_\Omega + (\Psi(u), v)_\Omega = (f, v)_\Omega \quad \forall\, v \in H_0^1(\Omega). \tag{3.6.3}$$

Under above assumptions, the existence and uniqueness of a solution $u \in H_0^1(\Omega)$ for (3.6.3) is shown in [69].

3.7 Shock-capturing virtual element method

Deducing an approximate solution for (3.6.1) in the singularly perturbed case $0 < \mathcal{D} \ll 1$, is interesting and requires suitable modification of (3.6.3). Under the effect of dominating convection and/or reaction phenomenon, layers are formed in the solutions and the variational form (3.6.3) produces solutions with unnecessary oscillations. The shock capturing (SC) technique added to SUPG method captures localised spurious oscillations in the crosswind direction. Thus we begin by presenting the SUPG and SC stabilized discrete formulation of (3.6.3) :Find $u_h \in V_h^p$ such that

$$A_{supg}(u_h, v_h) + A_{sc}(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h^p, \tag{3.7.1}$$

where,

$$A_{supg}(u_h, v_h) := (\mathcal{D}\,\nabla u_h, \nabla v_h)_\Omega + \Big(\overrightarrow{\beta}\cdot\nabla u_h, v_h\Big)_\Omega + (\Psi(u_h), v_h)_\Omega$$
$$+ \sum_{E\in\mathcal{T}_h} \tau_E \Big(-\mathcal{D}\Delta u_h + \overrightarrow{\beta}\cdot\nabla u_h + \Psi(u_h),\, \overrightarrow{\beta}\cdot\nabla v_h\Big)_E,$$
$$A_{sc}(u_h, v_h) := \sum_{E\in\mathcal{T}_h} (\,\xi_E(u_h)\, P^{sc}\,\nabla u_h,\, \nabla v_h\,)_E, \quad F(v_h) := (f, v_h)_\Omega + \sum_{E\in\mathcal{T}_h} \tau_E\big(f, \overrightarrow{\beta}\cdot\nabla v_h\big)_E.$$

The parameter τ_E is a local stabilization term associated with SUPG method. The variable $P^{sc} = (\,p^{sc}_{ij}\,)^2_{i,j=1}$ denotes a symmetric positive-definite (SPD) matrix function. Also ξ_E is a non-negative restricting function defined as (see [47]),

$$\xi_E(z) := \xi_E\big(\,r^*_E(z)\,\big) \quad \& \quad r^*_E(z) := \frac{\| -\nabla\cdot\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla z + \overrightarrow{\beta}\cdot\nabla z + \Psi(z) - f\|_{0,E}}{\|z\|_{1,E} + \sigma_E} \tag{3.7.2}$$

More assumptions and detailed definition of $\tau_E,\ P^{sc}$ and σ_E will be discussed in the sequel.

The terms in discrete scheme (3.7.1) are not computable in the VEM approach (see sec.3 in [70]). So with the help of projection operators defined in section 3.1.1, we proceed to appropriately redefine the scheme (3.7.1).

First we define $A_{vsg}(v, w) := a(v, w) + b(v, w) + c(v, w)$ where,

$$\begin{aligned}
a(v, w) \;&:=\; \big(\mathcal{D}\,\mathbf{\Pi}^0_{p-1}\nabla v,\, \mathbf{\Pi}^0_{p-1}\nabla w\big)_\Omega + \sum_{E\in\mathcal{T}_h}\tau_E\Big(\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla v,\ \overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla w\Big)_E \\
&\quad + \sum_{E\in\mathcal{T}_h}(\mathcal{D}_E + \tau_E\beta_E^2)\, S^E_1\big(\big(I - \Pi^\nabla_p\big)\, v, \big(I - \Pi^\nabla_p\big)\, w\big), \qquad (3.7.3)\\
b(v, w) \;&:=\; \frac{1}{2}\Big[\Big(\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla v,\, \Pi^0_p w\Big)_\Omega - \Big(\Pi^0_p v,\ \overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla w\Big)_\Omega\Big] \\
&\quad + \sum_{E\in\mathcal{T}_h}\tau_E\Big(-\nabla\cdot K\mathbf{\Pi}^0_{p-1}\nabla v,\ \overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla w\Big)_E, \qquad (3.7.4)\\
c(v, w) \;&:=\; \big(\hat{\Psi}(\Pi^0_p v),\, \Pi^0_p w\big)_\Omega + \sum_{E\in\mathcal{T}_h}\Psi_0\, S^E_2\big((I - \Pi^0_p)v, (I - \Pi^0_p)w\big) \\
&\quad + \sum_{E\in\mathcal{T}_h}\tau_E\big(\hat{\Psi}(\Pi^0_p v),\ \overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla w\big)_E. \qquad (3.7.5)
\end{aligned}$$

with $\mathcal{D}_E := \sup\limits_{x\in E}\mathcal{D}(x)$, $\mathcal{D}^\vee_E := \inf\limits_{x\in E}\mathcal{D}(x)$ and $\beta_E := \sup\limits_{x\in E}\|\overrightarrow{\beta}(x)\|_{0,\mathbb{R}^2}$.

The symmetric bilinear form S^E_1, S^E_2 in (3.7.3), (3.7.5) are functions defined on $V^p_E \times V^p_E$ ensuring that there exists non-zero positive constants $\alpha_*,\ \alpha^*,\ \mu_*,\ \mu^*$, with $\alpha_* \leq \alpha^*$ and

$\mu_* \le \mu^*$, independent of h_E, such that,

$$\begin{aligned} \alpha_*(\nabla u_h, \nabla u_h)_E &\le S_1^E(u_h, u_h) \le \alpha^*(\nabla u_h, \nabla u_h)_E \quad \forall u_h \in \ker(\Pi_p^\nabla), && (3.7.6)\\ \mu_*(u_h, u_h)_E &\le S_2^E(u_h, u_h) \le \mu^*(u_h, u_h)_E \qquad\quad \forall u_h \in \ker(\Pi_p^0). && (3.7.7) \end{aligned}$$

Let us denote, $\mathcal{L}z := -\nabla \cdot (\mathcal{D}\, \mathbf{\Pi}_{p-1}^0 \nabla z) + \overrightarrow{\beta} \cdot \mathbf{\Pi}_{p-1}^0 \nabla z + \hat{\Psi}(\Pi_p^0 z)$. Next we define,

$$\begin{aligned} A_{vsc}(z; v, w) := \sum_{E \in \mathcal{T}_h} \Big\{ \Big(\widehat{\xi}_E(z)\, P^{sc}\, \mathbf{\Pi}_{p-1}^0 \nabla v,\, \mathbf{\Pi}_{p-1}^0 \nabla w \Big)_E \\ + g_{sc}(z)\, S_1^E\left(\left(I - \Pi_p^\nabla\right) v, \left(I - \Pi_p^\nabla\right) w \right) \Big\}, \qquad (3.7.8) \end{aligned}$$

$$\text{with} \quad \widehat{\xi}_E(z) := \widehat{\xi}_E\big(R_E^*(z) \big) \text{ and } R_E^*(z) := \frac{\|\mathcal{L}z - f\|_{0,E}}{\|\Pi_p^0 z\|_{0,E} + |\Pi_p^\nabla z|_{1,E} + \sigma_E}. \qquad (3.7.9)$$

The variable σ_E is a regularisation parameter (see [47]) and the approximation $g_{sc}(\cdot) \in L^\infty(\Omega)$ in (3.7.8) is suitably chosen such that, there exists real constants $0 < \gamma_* \le \gamma^*$ satisfying

$$\gamma_*\, A_{sc}(z; w, w) \le A_{vsc}(z; w, w) \le \gamma^*\, A_{sc}(z; w, w) \quad \forall w \in V_h^p. \qquad (3.7.10)$$

For our analysis, we assume that the matrix norm $\|P^{sc}\|_{\infty,\Omega} \le 1$ and for each $E \in \mathcal{T}_h$, there exists a map $\phi_E : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{s \to 0} \phi_E(s) = 0 \quad \text{and} \quad 0 \le \widehat{\xi}_E(v) \le g_{sc}(v) \le \phi_E(h_E) \quad \forall v \in V_h^p. \qquad (3.7.11)$$

The computational choice for $g_{sc}(\cdot)$ will be discussed in the numerical experiment section.

Last we define, $$F_{vsg}(w) := \big(f, \Pi_p^0 w\big)_\Omega + \sum_{E \in \mathcal{T}_h} \tau_E \left(f, \overrightarrow{\beta} \cdot \mathbf{\Pi}_{p-1}^0 \nabla w \right)_E. \qquad (3.7.12)$$

Then, a discrete virtual element formulation of a general shock capturing scheme combined with the SUPG stabilization is : Find $u_h \in V_h^p$ such that

$$A_{vsg}(u_h, v_h) + A_{vsc}(u_h; u_h, v_h) = F_{vsg}(v_h) \qquad \forall v_h \in V_h^p. \qquad (3.7.13)$$

Remark 3.5. From the definition of $\widehat{\xi}_E(z)$ in (3.7.9) we note that the contribution of $A_{sc}(z; \cdot, \cdot)$ is restricted to those elements $E \in \mathcal{T}_h$ where the residual ($\mathcal{L}z - f$) is significant. The variants of the shock capturing method are determined by the choice of definition for P^{sc} in (3.7.8). In this paper we will discuss two variants of SC technique in section 3.9.

Remark 3.6. If sometimes Ψ' is not bound above, then we can consider the modification (2.18) in [47] for Ψ in the subsequent analysis. Noting that Ψ' is continuous and bounded on compact intervals of u implies Ψ is Lipschitz continuous, say with some Lipschitz constant $L_\Psi > 0$.

3.8 Preliminary Analysis

We introduce the norm

$$|||v|||^2 := \sum_{E\in\mathcal{T}_h} \Big(\|\sqrt{\mathcal{D}}\,\nabla v\|_{0,E}^2 + \Psi_0\,\|v\|_{0,E}^2 + \tau_E\,\|\overrightarrow{\beta}.\nabla v\|_{0,E}^2 \Big).$$

Consider the hp-inverse estimate (see (4.2) in [69]) satisfied by each $v_h \in V_h^p$,

$$\|\Delta v_h\|_{0,E} \le c_{\text{inv}}\, p^2\, h_E^{-1}\, |v_h|_{1,E}, \tag{3.8.1}$$

where, $c_{\text{inv}} > 0$ is independent of v, E, h_E and p. Now we state the coercivity result satisfied by $A_{vsg}(\cdot,\cdot)$.

Lemma 3.3. *Let us assume the following condition on τ_E for all $E \in \mathcal{T}_h$:*

$$0 \le \tau_E \le \frac{1}{4} min \left\{ \frac{h_E^2}{p^4 c_{inv}^2 \mathcal{D}_E},\ \frac{\Psi_0}{L_\Psi^2} \right\}. \tag{3.8.2}$$

Then we have $\quad A_{vsg}(w,w) \ge \theta\,|||w|||^2 \quad \forall w \in V_h^p,$ *where* $\theta = \min\left\{\frac{1}{2},\ \alpha_*,\ \mu_*\right\}$.

Proof. Let $w \in V_h^p$ be arbitrary. Consider $a(w,w)$ in (3.7.3). Using (3.7.6), the inequality $\|(I - \mathbf{\Pi}_{p-1}^0)\nabla u_h\|_{0,E} \le \|\nabla(I - \Pi_p^\nabla)u_h\|_{0,E}$ (see [12]) and definition of $\mathcal{D}_E$, we obtain

$$\begin{aligned} a(w,w) \ &\ge\ \sum_{E\in\mathcal{T}_h} \Big(\|\sqrt{\mathcal{D}}\,\mathbf{\Pi}_{p-1}^0 \nabla w\|_{0,E}^2 + \mathcal{D}_E\,\alpha_* \|(I-\mathbf{\Pi}_{p-1}^0)\nabla w\|_{0,E}^2 \\ &\qquad + \tau_E \|\overrightarrow{\beta}\cdot \mathbf{\Pi}_{p-1}^0 \nabla w\|_{0,E}^2 + \tau_E \beta_E^2\,\alpha_* \|(I-\mathbf{\Pi}_{p-1}^0)\nabla w\|_{0,E}^2 \Big) \\ &\ge\ \sum_{E\in\mathcal{T}_h} \Big(\|\sqrt{\mathcal{D}}\,\mathbf{\Pi}_{p-1}^0 \nabla w\|_{0,E}^2 + \alpha_* \|\sqrt{\mathcal{D}}(I-\mathbf{\Pi}_{p-1}^0)\nabla w\|_{0,E}^2 \\ &\qquad + \tau_E \|\overrightarrow{\beta}\cdot \mathbf{\Pi}_{p-1}^0 \nabla w\|_{0,E}^2 + \alpha_* \tau_E \|\overrightarrow{\beta}\cdot (I-\mathbf{\Pi}_{p-1}^0)\nabla w\|_{0,E}^2 \Big). \end{aligned} \tag{3.8.3}$$

Next we consider $b(w,w)$ in (3.7.4). Using Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|b(w,w)| &= \Big|0 + \sum_{E\in\mathcal{T}_h} \tau_E\Big(-\nabla\cdot\mathcal{D}\mathbf{\Pi}^0_{p-1}\nabla w,\ \overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla w\Big)_E\Big| \\
&\leq \sum_{E\in\mathcal{T}_h} \tau_E \|\nabla\cdot\mathcal{D}\mathbf{\Pi}^0_{p-1}\nabla w\|_{0,E}\,\|\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla w\|_{0,E}.
\end{aligned}
$$

Using the inverse estimate (3.8.1) and $0\leq\tau_E\leq\dfrac{1}{4}\dfrac{h_E^2}{p^4c_{\text{inv}}^2\mathcal{D}_E}$, we get,

$$
\|\nabla\cdot\mathcal{D}\mathbf{\Pi}^0_{p-1}\nabla w\|_{0,E} \leq \frac{1}{2\sqrt{\tau_E}}\|\sqrt{\mathcal{D}}\mathbf{\Pi}^0_{p-1}\nabla w\|_{0,E}. \tag{3.8.4}
$$

Then using (3.8.4) and Young's inequality for products we obtain,

$$
\begin{aligned}
|b(w,w)| &\leq \sum_{E\in\mathcal{T}_h}\Big(\frac{1}{\sqrt{2}}\|\sqrt{\mathcal{D}}\mathbf{\Pi}^0_{p-1}\nabla w\|_{0,E}\Big)\Big(\frac{\sqrt{\tau_E}}{\sqrt{2}}\|\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla w\|_{0,E}\Big) \\
&\leq \frac{1}{4}\sum_{E\in\mathcal{T}_h}\Big(\|\sqrt{\mathcal{D}}\mathbf{\Pi}^0_{p-1}\nabla w\|^2_{0,E} + \tau_E\|\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla w\|^2_{0,E}\Big)
\end{aligned}
$$

$$
\text{Thus, } b(w,w) \geq \frac{-1}{4}\sum_{E\in\mathcal{T}_h}\Big(\|\sqrt{\mathcal{D}}\mathbf{\Pi}^0_{p-1}\nabla w\|^2_{0,E} + \tau_E\|\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla w\|^2_{0,E}\Big). \tag{3.8.5}
$$

Next we consider $c(w,w)$ in (3.7.5). First we note, using $\Psi(0)=0$, Lipschitz continuity of Ψ and $\Psi'(s)\geq\Psi_0,\ s\in\mathbb{R}^+$, we obtain

$$
\big(\Psi(\Pi^0_p w),\Pi^0_p w\big) = \big(\Psi(\Pi^0_p w)-\Psi(0),\Pi^0_p w\big) \geq \big(\Psi_0\,\Pi^0_p w,\Pi^0_p w\big) \geq \frac{\Psi_0}{2}\sum_{E\in\mathcal{T}_h}\|\Pi^0_p w\|^2_{0,E}. \tag{3.8.6}
$$

Using the estimates (3.8.6) and (3.7.7) we get

$$
c(w,w) \geq \sum_{E\in\mathcal{T}_h}\big(\frac{\Psi_0}{2}\|\Pi^0_p w\|^2_{0,E} + \mu_*\Psi_0\|(I-\Pi^0_p)w\|^2_{0,E}\big) + \underbrace{\sum_{E\in\mathcal{T}_h}\tau_E\Big(\Psi(\Pi^0_p w),\ \overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla w\Big)_E}_{=I}.
$$

Applying Cauchy-Schwarz inequality, Lipschitz continuity of $\Psi(\cdot)$, $0\leq\tau_E\leq\frac{\Psi_0}{4L_g^2}$ and Young's inequality for products, we get,

$$
|I| \leq \frac{1}{4}\sum_{E\in\mathcal{T}_h}\Big(\Psi_0\|\Pi^0_p w\|^2_{0,E} + \tau_E\|\overrightarrow{\beta}\cdot\Pi^0_{p-1}\nabla w\|^2_{0,E}\Big).
$$

$$
\therefore\ c(w,w) \geq \sum_{E\in\mathcal{T}_h}\big(\frac{\Psi_0}{2}\|\Pi^0_p w\|^2_{0,E} + \mu_*\Psi_0\|(I-\Pi^0_p)w\|^2_{0,E} - \frac{1}{4}\tau_E\|\overrightarrow{\beta}\cdot\Pi^0_{p-1}\nabla w\|^2_{0,E}\big). \tag{3.8.7}
$$

Substituting the results (3.8.3), (3.8.5), (3.8.7) into $A_{vsg}(w,w) = a(w,w) + b(w,w) + c(w,w)$, we obtain $A_{vsg}(w,w) \geq \min\left\{\frac{1}{2},\, \alpha_*,\, \mu_*\right\} \; |||w||| $ for all $w \in V_h^p$. □

We shall show the existence of a discrete solution for the scheme (3.7.13) using the result stated in the following proposition (see [50]). Hereafter C denotes a generic positive constant independent of h_E, h, which takes different values at different incidents.

Proposition 3.1. *Let H be a finite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. Let $Q : H \to H$ be a continuous map. If there exists $k > 0$ such that, $\langle Q(w), w\rangle_H > 0$, $\forall w \in H$ with $\|w\|_H = k$, then $\exists$ a $z \in H$ such that $Q(z) = 0$ and $\|z\|_H \leq k$.*

Remark 3.7. For sake of completeness we recall remark 4.1 in [69] concisely. We define $\langle w_h, v_h\rangle_\star = \sum_{E\in\mathcal{T}_h} (\nabla w_h, \nabla v_h)_E \; \forall w_h, v_h \in V_h^p$, and $\|\cdot\|_\star := \langle\cdot,\cdot\rangle_\star^{\frac{1}{2}}$. Then, V_h^p with inner product $\langle\cdot,\cdot\rangle_\star$ is a finite dimensional Hilbert space. Also there exists constants $k_1,\, k_2 > 0$, such that, $\forall v_h \in V_h^p, \quad k_1\|v_h\|_\star \;\leq\; |||v_h||| \;\leq\; k_2\|v_h\|_\star$.

Theorem 3.2. *(Existence) Let the assumptions on* (3.6.1) *and* (3.8.2) *be satisfied. We assume the function $\widehat{\xi}(z)$ in* (3.7.8) *is continuous w.r.t z. Then there exists a solution $u_h \in V_h^p$ solving* (3.7.13) *and satisfying the inequality*

$$\|u_h\|_\star^2 + \gamma_* \sum_{E\in\mathcal{T}_h} \|\sqrt{\widehat{\xi}_E(u_h)}(P^{sc})^{\frac{1}{2}}\,\nabla u_h\|_{0,E}^2 \leq C\,\|f\|_0, \tag{3.8.8}$$

where γ_* is the constant established in (3.7.10)*.*

Proof. Using Riesz representation theorem, we consider a well-defined mapping $Q : V_h^p \to V_h^p$ defined such that

$$\langle Q(w_h), v_h\rangle_\star = A_{vsg}(w_h, v_h) + A_{vsc}(w_h; w_h, v_h) - F_{vsg}(v_h) \quad \forall v_h \in V_h^p. \tag{3.8.9}$$

We show that Q is a continuous map on V_h^p. For arbitrary $z_h, y_h \in V_h^p$, let us denote $\chi := z_h - y_h$ and $\mathcal{Q}_\chi := Q(z_h) - Q(y_h)$. Then,

$$\|\mathcal{Q}_\chi\|_\star^2 \;=\; \underbrace{A_{vsg}(z_h,\mathcal{Q}_\chi) - A_{vsg}(y_h,\mathcal{Q}_\chi)}_{=I} + \underbrace{A_{vsc}(z_h; z_h,\mathcal{Q}_\chi) - A_{vsc}(y_h; y_h,\mathcal{Q}_\chi)}_{=II}. \tag{3.8.10}$$

Using (3.6.2) and Theorem 4.1 of [69] , we have a constant $C > 0$ such that

$$I \;:=\; A_{vsg}(z_h,\mathcal{Q}_\chi) - A_{vsg}(y_h,\mathcal{Q}_\chi) \;\leq\; C\,\|z_h - y_z\|_\star\,\|\mathcal{Q}_\chi\|_\star. \tag{3.8.11}$$

Next, $II = \underbrace{A_{vsc}(z_h; z_h, \mathcal{Q}_\chi) - A_{vsc}(z_h; y_h, \mathcal{Q}_\chi)}_{=II_1} + \underbrace{A_{vsc}(z_h; y_h, \mathcal{Q}_\chi) - A_{vsc}(y_h; y_h, \mathcal{Q}_\chi)}_{=II_2}.$

Now, consider

$$II_1 = \sum_{E\in\mathcal{T}_h} \Big\{ \Big(\widehat{\xi}_E(z_h)\, P^{sc}\, \mathbf{\Pi}^0_{p-1}\nabla\chi,\, \mathbf{\Pi}^0_{p-1}\nabla\mathcal{Q}_\chi \Big)_E + g_{sc}(z_h)\, S_1^E\left(\left(I-\Pi_p^\nabla\right)\chi, \left(I-\Pi_p^\nabla\right)\mathcal{Q}_\chi\right) \Big\}.$$

Using the continuity of $\widehat{\xi}$, Cauchy-Schwarz inequality and (3.7.6) we get

$$\begin{aligned} II_1 &\le C\,\|z_h\|_\star\,\|\chi\|_\star\,\|\mathcal{Q}_\chi\|_\star + \|g_{sc}(z_h)\|_{\infty,\Omega}\,\alpha^*\,\|\chi\|_\star\,\|\mathcal{Q}_\chi\|_\star. \\ &\le C\,\|\chi\|_\star\,\|\mathcal{Q}_\chi\|_\star = C\,\|z_h-y_h\|_\star\,\|\mathcal{Q}_\chi\|_\star. \end{aligned} \tag{3.8.12}$$

Using the generalised Hölder's inequality (with $\frac{1}{\infty} = 0$), continuity of $\widehat{\xi}$ and Poincar$\acute{e}$ inequality we obtain

$$\begin{aligned} II_2 &= \sum_{E\in\mathcal{T}_h} \Big(\big[\,\widehat{\xi}_E(z_h) - \widehat{\xi}_E(y_h)\,\big]\, P^{sc}\,\mathbf{\Pi}^0_{p-1}\nabla y_h,\, \mathbf{\Pi}^0_{p-1}\nabla\mathcal{Q}_\chi \Big)_E \\ &\le C\,\|\widehat{\xi}_E(z_h) - \widehat{\xi}_E(y_h)\|_{0,\Omega}\,\|\mathbf{\Pi}^0_{p-1}\nabla y_h\|_{\infty,\Omega}\,\|\mathbf{\Pi}^0_{p-1}\nabla\mathcal{Q}_\chi\|_{0,\Omega} \\ &\le C\,\|z_h-y_h\|_{0,\Omega}\,\|\mathbf{\Pi}^0_{p-1}\nabla y_h\|_{\infty,\Omega}\,\|\mathcal{Q}_\chi\|_\star \le C\,\|z_h-y_h\|_\star\,\|\mathcal{Q}_\chi\|_\star. \end{aligned} \tag{3.8.13}$$

Substituting the results (3.8.11),(3.8.12),(3.8.13) into (3.8.10) we get

$$\|Q(z_h) - Q(y_h)\|_\star \;\le\; C\,\|z_h-y_h\|_\star \quad \forall z_h,\, y_h \in V_h^p. \tag{3.8.14}$$

Thus continuity of Q is established.

Next we bound $F_{vsg}(\cdot)$ in (3.7.12) in terms of $|||\cdot|||$ as follows,

$$\begin{aligned} F_{vsg}(v_h) &= \sum_{E\in\mathcal{T}_h} \Big((f, \Pi_p^0 v_h)_E + \tau_E \Big(f, \overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\Big)_E \Big) \\ &\le \sum_{E\in\mathcal{T}_h} \big[\, \|f\|_{0,E}\|\Pi_p^0 v_h\|_{0,E} + \tau_E\|f\|_{0,E}\|\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{0,E}\,\big] \\ &\le \sum_{E\in\mathcal{T}_h} \frac{\|f\|_{0,E}}{\Psi_0}\Psi_0\,\|v_h\|_{0,E} + \textstyle\sum_{E\in\mathcal{T}_h}\tau_E\|f\|_{0,E}\dfrac{\beta_E}{\sqrt{\mathcal{D}_E^\vee}}\|\sqrt{\mathcal{D}}\nabla v_h\|_{0,E}. \\ &\le C_\rho\,\|f\|_{0,\Omega}\,|||v_h|||, \quad (\text{ Use Hölders' ineq. } \&\ \tau_E \le \frac{\Psi_0}{L_\Psi^2}\,) \end{aligned} \tag{3.8.15}$$

where $C_\rho = (1/\Psi_0) + (\,\Psi_0/L_\Psi^2\,)\big\{ \max_{E\in\mathcal{T}_h}\big(\beta_E/\sqrt{\mathcal{D}_E^\vee}\big)$.

Using a property of SPD matrix P^{sc}, Lemma 3.3 and (3.8.15), for $v_h \in V_h^p$ we get,

$$\langle Q(v_h), v_h\rangle_\star = A_{vsg}(v_h, v_h) + A_{vsc}(v_h; v_h, v_h) - F_{vsg}(v_h) \geq \theta |||v_h|||^2 - C_\rho \|f\|_{0,\Omega} |||v_h|||,$$

where $\theta = \min\{1/2, \alpha_*, \mu_*\}$. Using the inequality $\frac{m}{\sqrt{\alpha}}\sqrt{\alpha}n \leq \frac{m^2}{\alpha} + \alpha n^2$ (choosing $\alpha = \frac{\theta}{2}$) for the term $C_\rho\|f\|_{0,\Omega} |||v_h|||$ and Remark 3.7 we get,

$$\langle Q(v_h), v_h\rangle_\star \geq \frac{\theta}{2}|||v_h|||^2 - \frac{2C_\rho^2}{\theta}\|f\|_{0,\Omega}^2 \geq \frac{k_1^2\theta}{2}\|v_h\|_\star^2 - \frac{2C_\rho^2}{\theta}\|f\|_{0,\Omega}^2. \tag{3.8.16}$$

For given f and a constant $C_\mathcal{B} = \frac{2C_\rho}{k_1\theta}$, let $\mathcal{B} = \{w_h \in V_h^p : \|w_h\|_\star > C_\mathcal{B}\|f\|_{0,\Omega}\}$. Thus using the estimate (3.8.16) we conclude that $\langle Q(v_h), v_h\rangle_\star > 0$ for all $v_h \in \mathcal{B}$. Now (thanks to remark 3.7) the proposition 3.1 guarantees that there exists $u_h \in V_h^p \setminus \mathcal{B}$ such that $Q(u_h) = 0$ and hence implies u_h is a solution for the discrete scheme (3.7.13). Using (3.8.9), (3.7.10) and (3.8.16), we also note

$$\langle Q(u_h), u_h\rangle_\star \geq \gamma_* A_{sc}(u_h; u_h, u_h) + \frac{\theta}{2}|||u_h|||^2 - \frac{2C_\rho^2}{\theta}\|f\|_{0,\Omega}^2. \tag{3.8.17}$$

In (3.8.17) using $Q(u_h) = 0$ we obtain the desired estimate (3.8.8). □

Remark 3.8. The following estimates involving operators Π_p^0, Π_p^∇ discussed in [12] will be used throughout in our analysis.

For $v_h \in V_h^p$ any $E \in \mathcal{T}_h$,

$$\|\mathbf{\Pi}_{p-1}^0 \nabla v_h\|_{0,E} \leq \|\nabla v_h\|_{0,E}. \tag{3.8.18}$$

$$\|\Pi_p^0 v_h\|_{0,E} \leq \|v_h\|_{0,E}. \tag{3.8.19}$$

$$\|\nabla(I - \Pi_p^\nabla)v_h\|_{0,E} \leq \|\nabla v_h\|_{0,E}. \tag{3.8.20}$$

$$\|(I - \Pi_p^0)v_h\|_{0,E} \leq \|v_h\|_{0,E}. \tag{3.8.21}$$

Lemma 3.4. *Let* τ_E *satisfy* (3.8.2)*. Consider* $w \in H_0^1(\Omega)$ *with* $(\nabla\cdot(\mathcal{D}\nabla w))|_E \in L^2(E)$*. Then for all* $v_h \in V_h^p$*, we have* $a(w, v_h) + b(w, v_h) \leq C\,\Theta(w)\,|||v|||$*, where,*

$$\Theta(w) := \Bigg\{ \Big[(1+\alpha^*) \max_{E\in\mathcal{T}_h} \Big(\frac{\mathcal{D}_E + \tau_E\beta_E^2}{\mathcal{D}_E^\vee}\Big) + \max_{E\in\mathcal{T}_h} \frac{\beta_E\sqrt{\mathcal{D}_E\tau_E}}{2\mathcal{D}_E^\vee} + \max_{E\in\mathcal{T}_h}\Big(\frac{\beta_E}{\sqrt{\mathcal{D}_E^\vee\Psi}}\Big)\Big] |||w||| + \Big(\sum_{E\in\mathcal{T}_h} \min\Big\{\frac{1}{\tau_E}; \frac{\beta_E^2}{\mathcal{D}_E^\vee}\Big\}\|w\|_{0,E}^2\Big)^{\frac{1}{2}} \Bigg\} \tag{3.8.22}$$

Proof. Consider the term $a(w, v_h)$. Using Cauchy-Schwarz inequality, (3.7.6), (3.8.18), (3.8.20), H$\ddot{o}$lder's inequality and definition of $|||\cdot|||$, we obtain

$$\begin{aligned}
a(w,v_h) &\le \sum_{E\in\mathcal{T}_h}\Big(\frac{\mathcal{D}_E}{\mathcal{D}_E^{\vee}}\,\|\sqrt{\mathcal{D}}\nabla w\|_{0,E}\,\|\sqrt{\mathcal{D}}\nabla v_h\|_{0,E}+\frac{\tau_E\beta_E^2}{\mathcal{D}_E^{\vee}}\|\sqrt{\mathcal{D}}\nabla w\|_{0,E}\,\|\sqrt{\mathcal{D}}\nabla v_h\|_{0,E}\Big)\\
&\qquad + \sum_{E\in\mathcal{T}_h}\Big(\Big(\frac{\mathcal{D}_E+\tau_E\beta_E^2}{\mathcal{D}_E^{\vee}}\Big)\alpha^*\|\sqrt{\mathcal{D}}\nabla w\|_{0,E}\,\|\sqrt{\mathcal{D}}\nabla v_h\|_{0,E}\Big)\\
&\le (1+\alpha^*)\max_{E\in\mathcal{T}_h}\Big(\frac{\mathcal{D}_E+\tau_E\beta_E^2}{\mathcal{D}_E^{\vee}}\Big)\sum_{E\in\mathcal{T}_h}\|\sqrt{\mathcal{D}}\nabla w\|_{0,E}\,\|\sqrt{\mathcal{D}}\nabla v_h\|_{0,E}\\
&\le (1+\alpha^*)\max_{E\in\mathcal{T}_h}\Big(\frac{\mathcal{D}_E+\tau_E\beta_E^2}{\mathcal{D}_E^{\vee}}\Big)\Big(\sum_{E\in\mathcal{T}_h}\|\sqrt{\mathcal{D}}\nabla w\|_{0,E}^2\Big)^{\frac{1}{2}}\Big(\sum_{E\in\mathcal{T}_h}\|\sqrt{\mathcal{D}}\nabla v_h\|_{0,E}^2\Big)^{\frac{1}{2}}\\
&\le (1+\alpha^*)\max_{E\in\mathcal{T}_h}\Big(\frac{\mathcal{D}_E+\tau_E\beta_E^2}{\mathcal{D}_E^{\vee}}\Big)\,|||w|||\,|||v_h|||. \qquad (3.8.23)
\end{aligned}$$

Next we estimate the term $b(w, v_h)$. Using triangle inequality and Cauchy-Schwarz inequality, we get,

$$\begin{aligned}
b(u,v_h) &\le \frac{1}{2}\sum_{E\in\mathcal{T}_h}\|\overrightarrow{\beta}\cdot\mathbf{\Pi}_{p-1}^0\nabla w\|_{0,E}\,\|\Pi_p^0 v_h\|_{0,E}+\frac{1}{2}\sum_{E\in\mathcal{T}_h}\|\Pi_p^0 w\|_{0,E}\,\|\overrightarrow{\beta}\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\|_{0,E}\\
&\quad+\sum_{E\in\mathcal{T}_h}\tau_E\,\|\nabla\cdot\mathcal{D}\mathbf{\Pi}_{p-1}^0\nabla w\|_{0,E}\,\|\overrightarrow{\beta}\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\|_{0,E}:=I_1+I_2+I_3. \quad (3.8.24)
\end{aligned}$$

Using (3.8.18), (3.8.19) and H$\ddot{o}$lder's inequality we get,

$$I_1 \le \sum_{E\in\mathcal{T}_h}\frac{\beta_E}{\sqrt{\mathcal{D}_E^{\vee}}}\|\sqrt{\mathcal{D}}\nabla w\|_{0,E}\frac{\sqrt{\Psi}}{\sqrt{\Psi}}\|v_h\|_{0,E}\le\max_{E\in\mathcal{T}_h}\Big(\frac{\beta_E}{\sqrt{\mathcal{D}_E^{\vee}\Psi}}\Big)|||w|||\,|||v_h|||. \qquad (3.8.25)$$

We derive two distinct bounds for the term I_2. First, using (3.8.18), (3.8.19), we have,

$$I_2\le\sum_{E\in\mathcal{T}_h}\|w\|_{0,E}\Big(\frac{\beta_E}{\sqrt{\mathcal{D}_E^{\vee}}}\Big)\|\sqrt{\mathcal{D}}\nabla v_h\|_{0,E}\le\left(\sum_{E\in\mathcal{T}_h}\frac{\beta_E^2}{\mathcal{D}_E^{\vee}}\,\|w\|_{0,E}^2\right)^{\frac{1}{2}}|||v_h|||. \qquad (3.8.26)$$

Second, using (3.8.18), (3.8.19), H$\ddot{o}$lder's inequality and $\tau_E\le(\Psi_0/L_\Psi^2)$ we get

$$\begin{aligned}
I_2 &= \sum_{E\in\mathcal{T}_h}\frac{1}{\sqrt{\tau_E}}\|w\|_{0,E}\sqrt{\tau_E}\frac{\beta_E}{\sqrt{\mathcal{D}_E^{\vee}}}\|\sqrt{\mathcal{D}}\nabla v_h\|_{0,E}\\
&\le \Big(\sum_{E\in\mathcal{T}_h}\frac{1}{\tau_E}\|w\|_{0,E}^2\Big)^{\frac{1}{2}}\Big(\sum_{E\in\mathcal{T}_h}\Big(\frac{\tau_E\beta_E^2}{\mathcal{D}_E^{\vee}}\Big)\|\sqrt{\mathcal{D}}\nabla v_h\|_{0,E}^2\Big)^{\frac{1}{2}}\le\Big(\sum_{E\in\mathcal{T}_h}\frac{1}{\tau_E}\|w\|_{0,E}^2\Big)^{\frac{1}{2}}\widetilde{C}|||v_h|||, \\
&\qquad (3.8.27)
\end{aligned}$$

where $\widetilde{C} = \max\limits_{E\in\mathcal{T}_h}\left(\dfrac{\Psi_0\,\beta_E^2}{L_\Psi^2\,\mathcal{D}_E^\vee}\right)^{\frac{1}{2}}$. Thus, combining (3.8.26) and (3.8.27) we obtain,

$$I_2 \;\leq\; \Big(\sum_{E\in\mathcal{T}_h}\min\Big\{\frac{1}{\tau_E};\,\frac{\beta_E^2}{\mathcal{D}_E^\vee}\Big\}\|w\|_{0,E}^2\Big)^{\frac{1}{2}}\;\max\{1,\,\widetilde{C}\}\;|\!|\!|v_h|\!|\!|. \tag{3.8.28}$$

Using (3.8.4), (3.8.18) and Hölder's inequality we get

$$\begin{aligned}
I_3 \;&\leq\; \sum_{E\in\mathcal{T}_h}\|\sqrt{\mathcal{D}}\boldsymbol{\Pi}_{p-1}^0\nabla w\|_{0,E}\frac{\sqrt{\tau_E}}{2}\,\beta_E\|\Pi_{p-1}^0\nabla v_h\|_{0,E}\\
&\leq\; \sum_{E\in\mathcal{T}_h}\frac{\beta_E\,\sqrt{\mathcal{D}_E\tau_E}}{2\mathcal{D}_E^\vee}\,\|\sqrt{\mathcal{D}}\nabla w\|_{0,E}\,\|\sqrt{\mathcal{D}}\nabla v_h\|_{0,E}\\
&\leq\; \left(\max_{E\in\mathcal{T}_h}\frac{\beta_E\,\sqrt{\mathcal{D}_E\tau_E}}{2\mathcal{D}_E^\vee}\right)\;|\!|\!|w|\!|\!|\;|\!|\!|v_h|\!|\!|.
\end{aligned} \tag{3.8.29}$$

Adding the results in (3.8.23), (3.8.25), (3.8.28) and (3.8.29) proves the claim. □

Hereafter, we assume each $E \in \mathcal{T}_h$ is convex. The following hp-virtual interpolation estimate (Lemma 4.4 in [69]) is useful in our analysis.

Proposition 3.2. *For $E \in \mathcal{T}_h$ and $u \in H_0^1(\Omega)\cap H^{\ell+1}(E)$, $\ell \in \mathbb{N}$, then there exists a $u_I \in V_h^p$, $m = min(p,\ell)$, satisfying,*

$$\|u-u_I\|_{0,E} + \frac{h_E}{p}\,|u-u_I|_{1,E} \leq C\frac{h_E^{m+1}}{p^{\ell+1}}\|u\|_{\ell+1,E}, \tag{3.8.30}$$

Now we prove a convergence result concerning the family of discrete solutions $\{\, u_h \in V_h^p \;:\; \forall\, h>0 \;\text{ and }\; u_h \;\text{ satisfies }\; (3.7.13)\,\}$.

Theorem 3.3. *Consider the assumptions of theorem 3.2 and* (3.7.11). *Suppose that* $\Psi_0\,\theta > 12(L_\Psi + \mu^*\Psi_0)$. *Let $u \in H_0^1(\Omega)$ be the exact solution of* (3.6.1) *with $u|_E \in H^{\ell+1}(E),\; p \geq \ell \geq 1$ for all $E \in \mathcal{T}_h$. Then any sequence $\{u_h\}_h$ of solution of* (3.7.13) *converges strongly to u in $H_0^1(\Omega)$, that is,*

$$\lim_{h\to 0}|\!|\!|u-u_h|\!|\!| = 0.$$

Proof. Consider the VEM discretisation of (3.6.3) : Find $U_h^* \in V_h^p$ such that

$$a_1(U_h^*,\,w_h) + a_2(U_h^*,\,w_h) + a_3(U_h^*,\,w_h) = \big(f,\,\Pi_p^0 w_h\big)_\Omega \quad \forall w_h \in V_h^p, \tag{3.8.31}$$

where, we define the terms $a_1(\cdot,\cdot)$, $a_2(\cdot,\cdot)$ and $a_3(\cdot,\cdot)$, as follows.

$$
\begin{aligned}
a_1(v,\, w) &= \big(\mathcal{D}\,\mathbf{\Pi}^0_{p-1}\nabla v,\, \mathbf{\Pi}^0_{p-1}\nabla w\big)_\Omega + \sum_{E\in\mathcal{T}_h} \mathcal{D}_E S^E_1\left(\left(I-\Pi^\nabla_p\right)v, \left(I-\Pi^\nabla_p\right)w\right).\\
a_2(v,\, w) &= (1/2)\Big[\Big(\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla v,\, \Pi^0_p w\Big)_\Omega - \Big(\Pi^0_p v,\, \overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla w\Big)_\Omega\Big].\\
a_3(v,\, w) &= \big(\hat{\Psi}(\Pi^0_p v),\, \Pi^0_p w\big)_\Omega + \sum_{E\in\mathcal{T}_h}\Psi_0\, S^E_2\left((I-\Pi^0_p)v, (I-\Pi^0_p)w\right).
\end{aligned}
$$

Let $U^*_h \in V^p_h$ satisfy problem (3.8.31). Define

$$
e := u - u_h = (u - U^*_h) + (U^*_h - u_h) = \eta_1 + \eta_2.
$$

First let us derive a bound for η_1. Let $u_I \in V^p_h$ is the virtual interpolant of u satisfying (3.8.30), $\vartheta_1 := u - u_I$ and $\vartheta_2 := U^*_h - u_I$. Then we note $\eta_1 = \vartheta_1 - \vartheta_2$. Note that both u and U^*_h satisfies (3.8.31). Hence,

$$
a_1(\eta_1, v_h) + b(\eta_1, v_h) + a_3(u, v_h) - a_3(U^*_h, v_h) = 0 \quad \forall v_h \in V^p_h. \tag{3.8.32}
$$

Now using Lemma 3.3 with $\tau_E = 0,\ \forall\, E\in\mathcal{T}_h$ and (3.8.32), we bound $\vartheta_2 \in V^p_h$ as follows,

$$
\begin{aligned}
\theta\,|||\vartheta_2||| &\leq a_1(\vartheta_1-\eta_1,\vartheta_2) + a_2(\vartheta_1-\eta_1,\vartheta_2) + a_3(\vartheta_2,\vartheta_2)\\
&\leq a_1(\vartheta_1,\vartheta_2) + a_2(\vartheta_1,\vartheta_2) + (a_3(u,\vartheta_2) - a_3(U^*_h,\vartheta_2)) + a_3(\vartheta_2,\vartheta_2)\\
&\leq a_1(\vartheta_1,\vartheta_2) + a_2(\vartheta_1,\vartheta_2) + \big(\Psi(\Pi^0_p u) - \Psi(\Pi^0_p U^*_h),\, \Pi^0_p\vartheta_2\big)_\Omega\\
&\quad + \sum_{E\in\mathcal{T}_h}\Psi_0\, S^E_2\left((I-\Pi^0_p)\eta_1, (I-\Pi^0_p)\vartheta_2\right) + a_3(\vartheta_2,\vartheta_2).
\end{aligned}\tag{3.8.33}
$$

Using Lemma 3.4 with $\tau_E = 0,\ \forall\, E\in\mathcal{T}_h$ and the Young's inequality $\frac{m}{\sqrt{\alpha}}\sqrt{\alpha}n \leq \frac{m^2}{\alpha} + \alpha n^2$ (with $\alpha = \theta/2$), we get

$$
a_1(\vartheta_1,\vartheta_2) + a_2(\vartheta_1,\vartheta_2) \;\leq\; C\,\Theta(\vartheta_1)\,|||\vartheta_2||| \leq \frac{2}{\theta}C\,(\Theta(\vartheta_1))^2 + \frac{\theta}{2}|||\vartheta_2|||^2. \tag{3.8.34}
$$

Using Cauchy-Schwarz inequality, Lipschitz continuity of Ψ, (3.8.19) and $ab \leq a^2 + b^2$, we get

$$
\begin{aligned}
\big(\Psi(\Pi^0_p u) - \Psi(\Pi^0_p U^*_h),\, \Pi^0_p\vartheta_2\big)_\Omega &\leq L_\Psi\,\|\eta_1\|_{0,\Omega}\,\|\vartheta_2\|_{0,\Omega}\\
&\leq \frac{L_\Psi}{\Psi_0}|||\eta_1|||^2 + \frac{L_\Psi}{\Psi_0}|||\vartheta_2|||^2.
\end{aligned}\tag{3.8.35}
$$

Using (3.7.7), (3.8.21), Hölder's inequality and then inequality $ab \le a^2 + b^2$ we get

$$\begin{aligned}\sum_{E\in\mathcal{T}_h} \Psi_0\, S_2^E\left((I-\Pi_p^0)\eta_1, (I-\Pi_p^0)\vartheta_2\right) &\le \sum_{E\in\mathcal{T}_h} \Psi_0\,\mu^*\,\|\eta_1\|_{0,E}\,\|\vartheta_2\|_{0,E} \\ &\le \frac{\Psi_0\,\mu^*}{\Psi_0}|||\eta_1|||^2 + \frac{\Psi_0\,\mu^*}{\Psi_0}|||\vartheta_2|||^2. \quad (3.8.36)\end{aligned}$$

Estimating $a_3(\vartheta_2,\vartheta_2)$ similar to (3.8.35)-(3.8.36) we get

$$a_3(\vartheta_2,\vartheta_2) \le \frac{L_\Psi + \Psi_0\,\mu^*}{\Psi_0}|||\vartheta_2|||^2. \quad (3.8.37)$$

Substituting the results (3.8.34)-(3.8.37) into (3.8.33) and simplifying, we obtain

$$|||\vartheta_2|||^2 \le \Upsilon_1\left(\frac{2}{\theta}C\,(\Theta(\vartheta_1))^2 + \frac{L_\Psi + \Psi_0\,\mu^*}{\Psi_0}|||\eta_1|||^2\right), \quad (3.8.38)$$

where $\quad \Upsilon_1 := \dfrac{2\,\Psi_0}{\theta\,\Psi_0 - 4(L_\Psi + \mu^*\Psi_0)} > 0.$

Thus, using the estimate (3.8.38), we note

$$|||\eta_1|||^2 \le 2\,|||\vartheta_1|||^2 + 2\,|||\vartheta_2|||^2 \le \Upsilon_2\,\Upsilon_1\,(4/\theta)\,C\,(\Theta(\vartheta_1))^2 + 2\,\Upsilon_2\,|||\vartheta_1|||^2, \quad (3.8.39)$$

where $\quad \Upsilon_2 := \dfrac{\theta\,\Psi_0 - 4(L_\Psi + \mu^*\Psi_0)}{\theta\,\Psi_0 - 8(L_\Psi + \mu^*\Psi_0)} > 0.$

Absorbing the coefficient of $|||\vartheta_1|||$ into the coefficient of $\Theta(\vartheta_1)$ in (3.8.39), we get

$$|||\eta_1||| \le C\,\Theta(u - u_I). \quad (3.8.40)$$

From (3.8.22) with $\tau_E = 0,\ \forall E \in \mathcal{T}_h$ and using (3.8.30), we obtain

$$|||\eta_1||| \le C\left(|||u - u_I||| + \sup_{E\in\mathcal{T}_h}\frac{\beta_E}{\sqrt{\mathcal{D}_E^\vee}}\,\|u - u_I\|_{0,\Omega}\right) \le C\,h^\ell\,|u|_{\ell+1,\Omega}. \quad (3.8.41)$$

Second, we derive a bound for η_2. From Lemma 3.3, we obtain

$$\begin{aligned}\theta\,|||\eta_2||| &\le A_{vsg}(\eta_2,\eta_2) \le |A_{vsg}(\eta_2,\eta_2)| \\ &= |a(U_h^*,\eta_2) - a(u_h,\eta_2) + b(U_h^*,\eta_2) - b(u_h,\eta_2) + c(\eta_2,\eta_2)|.\end{aligned}$$

Expanding the terms, we obtain,

$$\begin{aligned}
\theta\,|||\eta_2||| \;\leq\;& |a_1(U_h^*,\eta_2)+a_2(U_h^*,\eta_2)-a(u_h,\eta_2)+b(u_h,\eta_2)+c(\eta_2,\eta_2)| \\
+\;& \underbrace{|\sum_{E\in\mathcal{T}_h}\tau_E\left(\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla U_h^*,\;\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla\eta_2\right)_E+\tau_E\beta_E^2\,S_1^E\left(\left(I-\Pi_p^\nabla\right)U_h^*,\left(I-\Pi_p^\nabla\right)\eta_2\right)|}_{=I_1} \\
+\;& \underbrace{|\sum_{E\in\mathcal{T}_h}\tau_E\left(-\nabla\cdot K\mathbf{\Pi}^0_{p-1}\nabla U_h^*,\;\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla\eta_2\right)_E|}_{=I_2}.
\end{aligned}$$

Note that $\eta_2\in V_h^p,\; u_h$ solves (3.7.13) and U_h^* satisfy the equation (3.8.31). Therefore,

$$a_1(U_h^*,\eta_2)+a_2(U_h^*,\eta_2) \;=\; (f,\,\Pi_p^0\eta_2)_\Omega-a_3(U_h^*,\eta_2), \tag{3.8.42}$$

$$a(u_h,\eta_2)+b(u_h,\eta_2) \;=\; F_{vsg}(\eta_2)-c(u_h,\eta_2)-A_{sc}(u_h;u_h,\eta_2). \tag{3.8.43}$$

Thus, substituting (3.8.42), (3.8.43) and expanding,

$$\begin{aligned}
\theta\,|||\eta_2||| \;\leq\;& |(f,\,\Pi_p^0\eta_2)_\Omega-a_3(U_h^*,\eta_2)-F_{vsg}(\eta_2)+c(u_h,\eta_2)+A_{sc}(u_h;u_h,\eta_2)| \\
&+|c(\eta_2,\eta_2)|+I_1+I_2 \\
\leq\;& |A_{sc}(u_h;u_h,\eta_2)|+|\sum_{E\in\mathcal{T}_h}\tau_E(f,\;\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla\eta_2)_E|+|c(\eta_2,\eta_2)|+I_1+I_2 \\
&\underbrace{+|(\Psi(\Pi_p^0u_h)-\Psi(\Pi_p^0U_h^*),\,\Pi_p^0\eta_2)_\Omega|+|\Psi_0\sum_{E\in\mathcal{T}_h}S_2^E\left(\left(I-\Pi_p^0\right)\eta_2,\left(I-\Pi_p^0\right)\eta_2\right)_E|}_{=II_1} \\
&\underbrace{+|\sum_{E\in\mathcal{T}_h}\tau_E\left(\Psi(u_h),\;\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla\eta_2\right)_E|}_{=II_2}.
\end{aligned} \tag{3.8.44}$$

Using (3.7.11), (3.7.6) and (3.8.21) we obtain

$$\begin{aligned}
|A_{sc}(u_h;u_h,\eta_2)| \;\leq\;& C\sum_{E\in\mathcal{T}_h}\phi_E(h_E)\|u_h\|_{1,E}\,\|\eta_2\|_{1,E} \\
\leq\;& C\max_{E\in\mathcal{T}_h}\phi_E(h_E)\|u_h\|_{1,\Omega}\,\|\eta_2\|_{1,\Omega}.
\end{aligned} \tag{3.8.45}$$

From (3.8.2) we have, $$\tau_E \;\leq\; \frac{h_E^2}{p^4c_{\text{inv}}^2\mathcal{D}_E}. \tag{3.8.46}$$

Using Cauchy-Schwarz's inequality and (3.8.46), we get,

$$|\sum_{E\in\mathcal{T}_h}\tau_E(f,\;\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla\eta_2)_E|\leq C\,h^2\,\|f\|_{0,\Omega}\,\|\eta_2\|_{1,\Omega}. \tag{3.8.47}$$

Using Cauchy-Schwarz's inequality, assumption $\Psi(0) = 0$, Lipschitz continuity of Ψ, (3.7.7), (3.8.20), (3.8.46) and Hölder's inequality, we obtain

$$|c(\eta_2, \eta_2)| \leq \frac{(L_\Psi + \Psi_0 \mu^*)}{\Psi_0} |||\eta_2|||^2 + C\, h^2 \, \|\eta_2\|_{1,\Omega}^2. \quad (3.8.48)$$

Applying Cauchy-Schwarz, (3.8.46), (3.7.6), (3.8.21) and Hölder's inequality, we get

$$I_1 \leq C\, h^2 \, \|U_h^*\|_{1,\Omega} \, \|\eta_2\|_{1,\Omega}. \quad (3.8.49)$$

Using Cauchy-Schwarz inequality, (3.8.1), (3.8.46) and Hölder's inequality, we have

$$I_2 \leq C\, h \, \|U_h^*\|_{1,\Omega} \, \|\eta_2\|_{1,\Omega}. \quad (3.8.50)$$

Proceeding similar to inequality (3.8.48), we obtain

$$II_1 + II_2 \leq \frac{(L_\Psi + \Psi_0 \mu^*)}{\Psi_0} |||\eta_2|||^2 + C\, h^2 \, \|u_h\|_{1,\Omega} \, \|\eta_2\|_{1,\Omega}. \quad (3.8.51)$$

Substituting the results (3.8.45), (3.8.47)-(3.8.51) into (3.8.44) and simplifying, we obtain

$$|||\eta_2|||^2 \leq \mathcal{C}\, \|\eta_2\|_{1,\Omega} \Big\{ \max_{E \in \mathcal{T}_h} \phi_E(h_E) \|u_h\|_{1,\Omega} + h^2 \|f\|_{0,\Omega} + h^2 \|\eta_2\|_{1,\Omega} + (h^2 + h) \|U_h^*\|_{1,\Omega} \Big\},$$

where, $\mathcal{C} = (C\, \Psi_0)/(\theta \Psi_0 - 2(L_\Psi + \theta \mu^*)) > 0$.
Using the equivalence of the norms, $|||\cdot|||$ and $\|\cdot\|_{1,\Omega}$ in the space V_h^p we get

$$|||\eta_2||| \leq \mathcal{C} \Big\{ \max_{E \in \mathcal{T}_h} \phi_E(h_E) \|u_h\|_{1,\Omega} + h^2 \|f\|_{0,\Omega} + h^2 \|\eta_2\|_{1,\Omega} + (h^2 + h) \|U_h^*\|_{1,\Omega} \Big\}, \quad (3.8.52)$$

Note that the norms $\|f\|_{0,\Omega},\ \|\eta_2\|_{1,\Omega},\ \|u_h\|_{1,\Omega},\ \|U_h^*\|_{1,\Omega},\ |u|_{\ell+1,\Omega}$ are constants and $|||u - u_h||| \leq |||\eta_1||| + |||\eta_2|||$. Under the assumption (3.7.11) and letting $h \to 0$ in estimates (3.8.41) and (3.8.52), we obtain the desired result $\lim_{h \to 0} |||u - u_h||| = 0$. □

3.9 Error Analysis

In this section, we analyse two class of shock-capturing method based on adding isotropic artificial diffusion and anisotropic artificial diffusion. Error estimates involving rate of convergence is derived for both the SC classes.

We present a hp-polynomial interpolation estimates based on the polynomial mappings

Π_p^0 and Π_p^∇. To this end, let us consider the following polynomial estimate proved in Lemma 4.2 in [51].

Proposition 3.3. *Consider $E \in \mathcal{T}_h$ and let $u \in H^{s+1}(E)$. Then for each $p \in \mathbb{N}$ there exists a projection operator $\Pi^E : H^{s+1}(E) \to \mathbb{P}_p(E), \quad \Pi^E(u) = u_\pi$ such that $0 \le l \le s+1$, $\lambda = min(p, s)$,*

$$|u - u_\pi|_{l,E} \le C \frac{h_E^{\lambda+1-l}}{p^{s+1-l}} \|u\|_{s+1,E}. \tag{3.9.1}$$

Lemma 3.5. *Let $p \in \mathbb{N}$. For all $E \in \mathcal{T}_h$ and any $u \in H^{s+1}(E)$, $s \le p$, there exists a constant C independent of E and u such that*

$$\|u - \Pi_p^0 u\|_{0,E} \le C \left(\frac{h_E}{p}\right)^{s+1} \|u\|_{s+1,E}, \tag{3.9.2}$$

$$|u - \Pi_p^\nabla u|_{1,E} \le C \left(\frac{h_E}{p}\right)^{s} \|u\|_{s+1,E}. \tag{3.9.3}$$

Proof. By the property of operator Π_p^0 and Cauchy-Schwarz inequality, we note that,

$$\begin{aligned} \|u - \Pi_p^0 u\|_{0,E}^2 &= \left(u - \Pi_p^0 u,\, u - \Pi_p^0 u\right)_E = \left(u - \Pi_p^0 u,\, u - u_\pi\right)_E \\ &\le \|u - \Pi_p^0 u\|_{0,E} \, \|u - u_\pi\|_{0,E}, \end{aligned}$$

where $u_\pi \in \mathbb{P}_P(E)$ be as in proposition 3.3. Therefore, $\|u - \Pi_p^0 u\|_{0,E} \le \|u - u_\pi\|_{0,E}$. Now applying (3.9.1), we get the desired estimate (3.9.2). Following along similar lines for the term $|u - \Pi_p^\nabla u|_{1,E}$, we obtain (3.9.3). □

3.9.1 Adding isotropic diffusion

Consider the term $A_{vsc}(\cdot;\cdot,\cdot)$ in (3.7.8). To add isotropic diffusion, we set the parameters P^{sc} and $\widehat{\xi}(\cdot)$ in (3.7.8) as follows :

$$P^{sc} := \mathbf{I} \quad \text{and} \quad \widehat{\xi}_E(z) := \rho_E(z)\,[R_E^*(z)]^2, \tag{3.9.4}$$

where $R_E^*(z)$ is as defined in (3.7.9).

In the error analysis we will encounter a term consisting of H^1 seminorm of the solution $u_h \in V_h^p$ of discrete scheme (3.7.13) and the virtual element interpolant $u_I \in V_h^p$ of the exact solution u of (3.6.1). In the following lemma we state this term and show that it is uniformly bounded with respect to h.

Lemma 3.6. *Consider the assumptions given in Theorem 3.3 and σ_E be as in* (3.7.9)*. Let $u_I \in V_h^p$ be the virtual element interpolant of the exact solution u of* (3.6.1) *and for $E \in \mathcal{T}_h$ denote $\mathcal{N}_E := \dfrac{|u_I|_{1,E}}{|u_h|_{1,E} + \sigma_E}$. Then there exists a constant C independent of the problem data and h, such that*

$$\max_{E\in\mathcal{T}_h} \mathcal{N}_E \leq C. \tag{3.9.5}$$

Proof. From the Theorem 3.3 and using the equivalence of norms in remark 3.7, we have that $\lim_{h\to 0} |u - u_h|_{1,\Omega} = 0$. Consequently, there exists $h_0 > 0$ such that $\forall\, h \leq h_0,$ we obtain,

$$|u - u_h|_{1,E} \leq (1/2)\, |u|_{1,E} \qquad \forall\, E \in \mathcal{T}_h. \tag{3.9.6}$$

For $E \in \mathcal{T}_h$, and using (3.2) we note,

$$|u_I|_{1,E} \leq |u|_{1,E} + |u - u_I|_{1,E} \leq C\, |u|_{1,\Omega}. \tag{3.9.7}$$

Now using the inequality $|a| - |b| \leq |a - b|$, (3.9.6) and (3.9.7), we obtain

$$\mathcal{N}_E \leq \frac{C\, |u|_{1,\Omega}}{||u|_{1,E} - |u - u_h|_{1,E}| + \sigma_E} \leq \frac{2\, C\, |u|_{1,\Omega}}{|u|_{1,E} + \sigma_E} \leq C, \tag{3.9.8}$$

for we note $|u|_{1,\Omega}$ is a constant, $\min_{E\in\mathcal{T}_h} \sigma_E \leq |u|_{1,E} + \sigma_E$ and hence C is a constant independent of the data and h. □

Theorem 3.4. *Let $u \in H_0^1(\Omega) \cap H^{\ell+1}(E)$, $\ell \in \mathbb{N}$ be the exact solution of* (3.6.1) *with $(\nabla \cdot \mathcal{D}\nabla u)|_E \in L^2(E)$ for all $E \in \mathcal{T}_h$. Suppose the stabilization parameter τ_E is such that*

$$0 \leq \tau_E \leq \frac{\theta}{16} min \left\{ \frac{h_E^2}{p^4 c_{inv}^2 \mathcal{D}_E}, \frac{\Psi_0}{L_\Psi^2} \right\}, \tag{3.9.9}$$

and let $\Psi_0\,(15\theta - 2\gamma^*) > 2(64 + 3\gamma^*)(L_\Psi + \mu^*)$ be satisfied. For a sufficiently small $\kappa > 0$, we suppose that $0 \leq \widehat{\xi}_E(w_h) \leq \kappa\, \tau_E, \quad \forall w_h \in V_h^p$. Then the solution $u_h \in V_h^p$ of (3.7.13) *with* (3.9.4) *satisfies,*

$$\begin{aligned} &|||u - u_h|||^2 + \gamma_* \sum_{E\in\mathcal{T}_h} \widehat{\xi}_E(u_h)\, |u - u_h|_{1,E}^2 \\ &\leq C \left\{ [\Theta(u - u_I)]^2 + \sum_{E\in\mathcal{T}_h} \left[\tau_E\, \beta_E^2 + \mathcal{D}_E \right] \left(\frac{h_E}{p}\right)^{2\ell} \|u\|_{\ell+1,E}^2 \right\}, \end{aligned} \tag{3.9.10}$$

where $u_I \in V_h^p$ is the virtual interpolant of u as in (3.8.30) *and constant $C > 0$ is dependent on $L_\Phi,\ \Psi_0,\ \theta,\ \gamma^*$, but independent of $\mathcal{D}$, h_E.*

Proof. Let us denote $\zeta_1 := u_h - u_I$ and $\zeta_2 := u - u_I$. Then $u - u_h = \zeta_2 - \zeta_1$.

From lemma 3.3 we obtain

$$\begin{aligned}\theta\,|||\zeta_1|||^2 + A_{vsc}(u_h;\zeta_1,\zeta_1) &\leq A_{vsg}(\zeta_1,\zeta_1) + A_{vsc}(u_h;\zeta_1,\zeta_1)\\ &\leq (a+b+c)(u_h-u_I,\ \zeta_1) + A_{vsc}(u_h;u_h,\zeta_1) - A_{vsc}(u_h;u_I,\zeta_1).\end{aligned}$$

Adding and substracting $A_{vsg}(u,\zeta_1),\ c(u_h,\zeta_1)$ and since u_h is a solution of (3.7.13), we get

$$\begin{aligned}\theta\,|||\zeta_1|||^2 + A_{vsc}(u_h;\zeta_1,\zeta_1) \ &\leq\ (a+b)(\zeta_2,\zeta_1) + c(\zeta_1,\zeta_1) + [c(u,\zeta_1) - c(u_h,\zeta_1)]\\ &+ \sum_{E\in\mathcal{T}_h}\tau_E\left(f - \mathcal{L}(u),\ \overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{k-1}\nabla\zeta_1\right)_E - A_{vsc}(u_h;u_I,\zeta_1).\end{aligned} \tag{3.9.11}$$

Using lemma 3.4 and the inequality $\dfrac{m}{\sqrt{\alpha}}\sqrt{\alpha}n \leq \dfrac{m^2}{\alpha} + \alpha n^2$ (choosing $\alpha = \dfrac{\theta}{16}$) we get

$$(a+b)(\zeta_2,\zeta_1) \ \leq\ \frac{16}{\theta}C(\Theta(\zeta_2))^2 + \frac{\theta}{16}|||\zeta_1|||^2. \tag{3.9.12}$$

Using Cauchy-Schwarz inequality, (3.7.7), $\Psi(0)$=0, Lipschitz continuity of Ψ, we get

$$\begin{aligned}c(\zeta_1,\zeta_1) \ &\leq\ \sum_{E\in\mathcal{T}_h}\Big\{\,\|\hat{\Psi}(\Pi^0_p\zeta_1) - \hat{\Psi}(0)\|_{0,E}\|\Pi^0_p\zeta_1\|_{0,E} + \mu^*\,\|(I-\Pi^0_p)\zeta_1\|^2_{0,E}\\ &\quad + \tau_E\,\|\hat{\Psi}(\Pi^0_p\zeta_1) - \hat{\Psi}(0)\|_{0,E}\,\|\overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla\zeta_1\|_{0,E}\,\Big\}\\ &\leq\ \sum_{E\in\mathcal{T}_h}\Big\{\,L_\Psi\,\|\zeta_1\|^2_{0,E} + \mu^*\,\|\zeta_1\|^2_{0,E} + \tau_E\,\beta_E\,L_\Psi\,\|\zeta_1\|_{0,E}\,\|\nabla\zeta_1\|_{0,E}\,\Big\}\\ &\qquad\qquad\qquad\qquad\text{(use (3.8.19), (3.8.21))}\\ &\leq\ \sum_{E\in\mathcal{T}_h}\Big\{\,\frac{L_\Psi+\mu^*}{\Psi_0}\,\Psi_0\,\|\zeta_1\|^2_{0,E} + \frac{\sqrt{\theta\Psi_0}}{4}\|\zeta_1\|_{0,E}\,\frac{\sqrt{\theta}}{4}\,\frac{\beta_E h_E}{p^2 c_{\text{inv}}\sqrt{\mathcal{D}_E}}\|\nabla\zeta_1\|_{0,E}\,\Big\}. \quad \text{(use (3.9.9))}\end{aligned}$$

Noting $p \geq 1,\ \dfrac{1}{\mathcal{D}_E} \leq \dfrac{1}{\mathcal{D}_E^\vee}$ and for $h_E \leq \dfrac{c_{\text{inv}}\mathcal{D}_E^\vee}{\beta_E}$, using Hölder's inequality, and then using Young's inequality for products, we obtain,

$$c(\zeta_1,\zeta_1) \ \leq\ \frac{L_\Psi+\mu^*}{\Psi_0}\,|||\zeta_1|||^2 + \frac{\theta}{8}|||\zeta_1|||^2. \tag{3.9.13}$$

Now similar to (3.9.13), we obtain the estimate

$$c(u,\zeta_1) - c(u_h,\zeta_1) \le \frac{L_\Psi + \mu^*}{\Psi_0}\,(\,|||u-u_h|||^2 + |||\zeta_1|||^2\,) + \frac{\theta}{16}\,(\,|||u-u_h|||^2 + |||\zeta_1|||^2\,). \qquad (3.9.14)$$

Now we estimate,

$$\begin{aligned} I &= \sum_{E\in\mathcal{T}_h} \tau_E \left(f - \mathcal{L}(u),\ \overrightarrow{\beta}\cdot \mathbf{\Pi}^0_{p-1}\nabla\zeta_1 \right)_E \\ &= \sum_{E\in\mathcal{T}_h} \tau_E \Big(\nabla\cdot K(\mathbf{\Pi}^0_{p-1}\nabla u - \nabla u) + \overrightarrow{\beta}\cdot(\nabla u - \mathbf{\Pi}^0_{p-1}\nabla u) + (\widehat{\Psi}(u) - \widehat{\Psi}(\Pi^0_p u)),\ \overrightarrow{\beta}\cdot \mathbf{\Pi}^0_{k-1}\nabla\zeta_1 \Big)_E \end{aligned}$$

Using Cauchy-Schwarz's inequality and triangle inequality we get

$$\begin{aligned} I \ \le\ & \sum_{E\in\mathcal{T}_h} \sqrt{\tau_E}\Big(\|\nabla\cdot\mathcal{D}\,(\mathbf{\Pi}^0_{p-1}\nabla u - \nabla u)\|_{0,E} + \beta_E\|\nabla u - \mathbf{\Pi}^0_{p-1}\nabla u\|_{0,E} \\ & \qquad + \|\widehat{\Psi}(u) - \widehat{\Psi}(\Pi^0_p u)\|_{0,E}\Big)\sqrt{\tau_E}\,\beta_E\|\mathbf{\Pi}^0_{k-1}\nabla\zeta_1\|_{0,E}. \end{aligned}$$

Using (3.8.4), Lipschitz continuity of Ψ and (3.8.19), we get

$$\begin{aligned} I \ \le\ & \sum_{E\in\mathcal{T}_h} \Big(\tfrac{1}{2}\|\sqrt{\mathcal{D}}(\mathbf{\Pi}^0_{p-1} - I)\nabla u\|_{0,E} + \sqrt{\tau_E}\beta_E\|(I-\mathbf{\Pi}^0_{p-1})\nabla u\|_{0,E} \\ & \qquad + \sqrt{\Psi_0}\|(I-\Pi^0_p)u\|_{0,E}\Big)\sqrt{\frac{\tau_E}{\mathcal{D}^\vee_E}}\,\beta_E\|\sqrt{\mathcal{D}}\nabla\zeta_1\|_{0,E}. \end{aligned}$$

Using the inequality $\|(I-\mathbf{\Pi}^0_{p-1})\nabla v_h\|_{0,E} \le \|\nabla(I-\Pi^\nabla_p)v_h\|_{0,E}$ (see [12]), (3.9.2), (3.9.3), Hölder's inequality and the inequality $\frac{m}{\sqrt{\alpha}}\sqrt{\alpha}n \le \frac{m^2}{\alpha} + \alpha n^2$ (choosing $\alpha = \frac{\theta}{16}$), we obtain

$$\begin{aligned} I \ \le\ & \sum_{E\in\mathcal{T}_h} \Big(\tfrac{\sqrt{\tau_E}\beta_E}{2}\sqrt{\tfrac{\mathcal{D}_E}{\mathcal{D}^\vee_E}}\|\nabla(I-\Pi^\nabla_p)u\|_{0,E} + \tfrac{\tau_E\beta_E^2}{\sqrt{\mathcal{D}^\vee_E}}\|\nabla(I-\Pi^\nabla_p)u\|_{0,E} \\ & \qquad + \sqrt{\frac{\tau_E}{\mathcal{D}^\vee_E}}\,\beta_E\sqrt{\Psi_0}\|u - \Pi^0_p u\|_{0,E}\Big)\|\sqrt{\mathcal{D}}\nabla\zeta_1\|_{0,E} \\ \le\ & \sum_{E\in\mathcal{T}_h} C\,\sqrt{\tau_E}\beta_E\,\big(\frac{h_E}{p}\big)^{\ell}\,\|u\|_{\ell+1,E}\,\|\sqrt{\mathcal{D}}\nabla\zeta_1\|_{0,E} \\ \le\ & \Big(\sum_{E\in\mathcal{T}_h} C\,\tau_E\,\beta_E^2\,\big(\frac{h_E}{p}\big)^{2\ell}\,\|u\|^2_{\ell+1,E}\Big)^{\frac{1}{2}}\Big(\sum_{E\in\mathcal{T}_h}\|\sqrt{\mathcal{D}}\nabla\zeta_1\|_{0,E}\Big)^{\frac{1}{2}} \\ \le\ & C\sum_{E\in\mathcal{T}_h}\tau_E\,\beta_E^2\,\big(\frac{h_E}{p}\big)^{2\ell}\,\|u\|^2_{\ell+1,E} + \frac{\theta}{16}|||\zeta_1|||^2. \end{aligned} \qquad (3.9.15)$$

Let $II := |A_{vsc}(u_h; u_I, \zeta_1)|$. Using Cauchy-Schwarz inequality, (3.7.6) and Hölder's inequality, we get

$$
\begin{aligned}
II &\leq \sum_{E\in\mathcal{T}_h} \|(\widehat{\xi}(u_h)P^{sc})^{\frac{1}{2}}\mathbf{\Pi}^0_{p-1}\nabla u_I\|_{0,E}\, \|(\widehat{\xi}(u_h)P^{sc})^{\frac{1}{2}}\mathbf{\Pi}^0_{p-1}\nabla \zeta_1\|_{0,E} \\
&\quad + \sum_{E\in\mathcal{T}_h} g_{sc}(u_h)^{\frac{1}{2}}\|\nabla(I-\Pi^{\nabla}_p)u_I\|_{0,E}\, g_{sc}(u_h)^{\frac{1}{2}}\|\nabla(I-\Pi^{\nabla}_p)\zeta_1\|_{0,E}. \\
&\leq \Big(\Upsilon_1 \sum_{E\in\mathcal{T}_h} \|(\widehat{\xi}(u_h)P^{sc})^{\frac{1}{2}}\mathbf{\Pi}^0_{p-1}\nabla u_I\|^2_{0,E}\Big)^{\frac{1}{2}} \Big(\Upsilon_1^{-1} \sum_{E\in\mathcal{T}_h} \|(\widehat{\xi}(u_h)P^{sc})^{\frac{1}{2}}\mathbf{\Pi}^0_{p-1}\nabla \zeta_1\|^2_{0,E}\Big)^{\frac{1}{2}} \\
&\quad + \Big(\Upsilon_1 \sum_{E\in\mathcal{T}_h} g_{sc}(u_h)\|\nabla(I-\Pi^{\nabla}_p)u_I\|^2_{0,E}\Big)^{\frac{1}{2}} \Big(\Upsilon_1^{-1} \sum_{E\in\mathcal{T}_h} g_{sc}(u_h)\|\nabla(I-\Pi^{\nabla}_p)\zeta_1\|^2_{0,E}\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Using Young's inequality for products, (3.8.20) and (3.7.6), we get

$$
\begin{aligned}
II &\leq \frac{2}{3}\Upsilon_2 \sum_{E\in\mathcal{T}_h} \Big(\|(\widehat{\xi}(u_h)P^{sc})^{\frac{1}{2}}\mathbf{\Pi}^0_{p-1}\nabla u_I\|^2_{0,E} + g_{sc}(u_h)\|\nabla(I-\Pi^{\nabla}_p)u_I\|^2_{0,E}\Big) \\
&\quad + \frac{3}{8}\Upsilon_2 \sum_{E\in\mathcal{T}_h} \Big(\|(\widehat{\xi}(u_h)P^{sc})^{\frac{1}{2}}\mathbf{\Pi}^0_{p-1}\nabla \zeta_1\|^2_{0,E} + g_{sc}(u_h)\|\nabla(I-\Pi^{\nabla}_p)\zeta_1\|^2_{0,E}\Big). \\
&\leq \frac{2}{3}\Upsilon_2 \sum_{E\in\mathcal{T}_h} \Big(\|(\widehat{\xi}(u_h)P^{sc})^{\frac{1}{2}}\mathbf{\Pi}^0_{p-1}\nabla u_I\|^2_{0,E} + \frac{1}{\alpha_*}\, g_{sc}(u_h)\, S_1^E((I-\Pi^{\nabla}_p)u_I, (I-\Pi^{\nabla}_p)u_I)\Big) \\
&\quad + \frac{3}{8}\Upsilon_2^{-1} \sum_{E\in\mathcal{T}_h} \Big(\|(\widehat{\xi}(u_h)P^{sc})^{\frac{1}{2}}\mathbf{\Pi}^0_{p-1}\nabla \zeta_1\|^2_{0,E} + \frac{1}{\alpha_*}\, g_{sc}(u_h)\, S_1^E((I-\Pi^{\nabla}_p)\zeta_1, (I-\Pi^{\nabla}_p)\zeta_1)\Big) \\
&\leq \frac{2}{3}\gamma^*\, A_{sc}(u_h; u_I, u_I) + \frac{3}{8}\, A_{vsc}(u_h; \zeta_1, \zeta_1). \quad (\text{ as } \Upsilon_2 \leq 1)
\end{aligned}
\tag{3.9.16}
$$

where $\Upsilon_1 := \dfrac{4}{3}\Upsilon_2$ and $\Upsilon_2 := \max\{1,\ \dfrac{1}{\alpha_*}\} = \min\{1,\ \alpha_*\} \leq 1.$

Next we evaluate the term $A_{sc}(u_h; u_I, u_I)$. Using the definition of A_{vsc} in (3.7.1) along with (3.7.2) and (3.9.4), we get

$$
\begin{aligned}
A_{sc}(u_h; u_I, u_I) &\leq \sum_{E\in\mathcal{T}_h} |\xi(u_h)\|\nabla u_I\|^2_{0,E} \leq \sum_{E\in\mathcal{T}_h} \rho_E(u_h)\, |R^*_E(u_h)|^2\, |u_I|^2_{1,E} \\
&\leq \kappa \sum_{E\in\mathcal{T}_h} \tau_E\, \| -\nabla\cdot\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla u_h + \overrightarrow{\beta}\cdot\nabla u_h + \Psi(u_h) - f\|^2_{0,E}\, \frac{|u_I|^2_{1,E}}{||u_h|_{1,E} + \sigma_E|^2} \\
&\leq \kappa \big(\max_{E\in\mathcal{T}_h}\mathcal{N}_E\big)^2 \sum_{E\in\mathcal{T}_h} \tau_E\, \| -\nabla\cdot\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla u_h + \overrightarrow{\beta}\cdot\nabla u_h + \Psi(u_h) - f\|^2_{0,E} \\
&\leq \kappa \big(\max_{E\in\mathcal{T}_h}\mathcal{N}_E\big)^2 \sum_{E\in\mathcal{T}_h} 4\,\tau_E \Big\{\| -\nabla\cdot\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla(u_h - u_I) + \overrightarrow{\beta}\cdot\nabla(u_h - u_I)\|^2_{0,E} \\
&\qquad + \| -\nabla\cdot\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla(u_I - u) + \overrightarrow{\beta}\cdot\nabla(u_I - u)\|^2_{0,E} \\
&\qquad + \| -\nabla\cdot(\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla u - \mathcal{D}\nabla u)\|^2_{0,E} + \|\Psi(u_h) - \Psi(u)\|^2_{0,E}\Big\}.
\end{aligned}
\tag{3.9.17}
$$

Using triangle inequality and (3.8.4), we get,

$$
\begin{aligned}
4\sum_{E\in\mathcal{T}_h} 4\tau_E \,\| -\nabla\cdot\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla(u_h-u_I) + \overrightarrow{\beta}\cdot\mathbf{\Pi}^0_{p-1}\nabla(u_h-u_I)\|^2_{0,E} \\
\leq 8\sum_{E\in\mathcal{T}_h}\tau_E\left\{\| -\nabla\cdot\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla\zeta_1\|^2_{0,E} + \|\overrightarrow{\beta}\cdot\nabla\zeta_1\|^2_{0,E}\right\} \\
\leq 8\sum_{E\in\mathcal{T}_h}\left\{\|\mathcal{D}\nabla\zeta_1\|^2_{0,E} + \tau_E\,\|\overrightarrow{\beta}\cdot\nabla\zeta_1\|^2_{0,E}\right\} \leq 8\,|||\zeta_1|||^2. \quad (3.9.18)
\end{aligned}
$$

Similarly, using (3.8.4) and the definition of ζ_2, we get,

$$
4\sum_{E\in\mathcal{T}_h}\tau_E\,\| -\nabla\cdot\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla(u_I-u) + \overrightarrow{\beta}\cdot\nabla(u_I-u)\|^2_{0,E} \leq 8\,|||\zeta_2|||^2. \quad (3.9.19)
$$

Using (3.8.4) and (3.9.3), we obtain,

$$
\begin{aligned}
4\sum_{E\in\mathcal{T}_h}\tau_E\,\| -\nabla\cdot(\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla u - \nabla u)\|^2_{0,E} &\leq \sum_{E\in\mathcal{T}_h} 4\,\mathcal{D}_E\|(\mathbf{\Pi}^0_{p-1}\nabla u - \nabla u)\|^2_{0,E} \\
&\leq \sum_{E\in\mathcal{T}_h} 4\,\mathcal{D}_E\|\nabla(\Pi^\nabla_p - I)u\|^2_{0,E} \\
&\leq C\sum_{E\in\mathcal{T}_h}\mathcal{D}_E\Big(\frac{h_E}{p}\Big)^{2\ell}\|u\|^2_{\ell+1,E}. \quad (3.9.20)
\end{aligned}
$$

Using Lipschitz continuity of Ψ, noting $0 \leq \tau_E \leq \dfrac{\theta}{64}\dfrac{\Psi_0}{L^2_\Psi}$ and $\theta < 1,$ we have,

$$
4\sum_{E\in\mathcal{T}_h}\tau_E\,\|\Psi(u_h)-\Psi(u)\|^2_{0,E} \leq \frac{\theta}{16}|||u-u_h|||^2 \leq |||u-u_h|||^2. \quad (3.9.21)
$$

The Lemma 3.6 implies for sufficiently small $\kappa > 0$, we have

$$
\kappa\Big(\max_{E\in\mathcal{T}_h}\mathcal{N}_E\Big)^2 \leq \frac{\theta}{64}. \quad (3.9.22)
$$

Substituting (3.9.18)-(3.9.22) into (3.9.16), we obtain

$$
\begin{aligned}
II \leq\; & \frac{\theta}{12}\gamma^*|||\zeta_1|||^2 + \frac{\theta}{12}\gamma^*|||\zeta_2|||^2 + C\sum_{E\in\mathcal{T}_h}\mathcal{D}_E\Big(\frac{h_E}{p}\Big)^{2\ell}\|u\|^2_{\ell+1,E} \\
& + \frac{\theta}{96}\gamma^*|||u-u_h|||^2 + \frac{3}{8}A_{vsc}(u_h;\zeta_1,\zeta_1). \quad (3.9.23)
\end{aligned}
$$

Substituting the estimates (3.9.12), (3.9.13), (3.9.14), (3.9.15) and (3.9.23), into (3.9.11), we obtain

$$\theta\,|||\zeta_1|||^2 \;+\; A_{vsc}(u_h;\zeta_1,\zeta_1) \;\le\; \frac{5\theta}{16}|||\zeta_1|||^2 + 2\frac{L_\Psi+\mu^*}{\Psi_0}\,|||\zeta_1|||^2 + \frac{\theta}{12}\gamma^*|||\zeta_1|||^2$$
$$+\frac{16}{\theta}C(\Theta(\zeta_2))^2 + \frac{\theta}{12}\gamma^*|||\zeta_2|||^2 + \Big(\frac{L_\Psi+\mu^*}{\Psi_0} + \frac{\theta}{16} + \frac{\theta}{96}\gamma^*\Big)\,|||u-u_h|||^2$$
$$+C\sum_{E\in\mathcal{T}_h}\big[\,\tau_E\,\beta_E^2 + \mathcal{D}_E\,\big]\,\big(\frac{h_E}{p}\big)^{2\ell}\,\|u\|^2_{\ell+1,E} + \frac{3}{8}A_{vsc}(u_h;\zeta_1,\zeta_1).$$

Noting that $\theta < 1$ and simplifying, we obtain

$$|||\zeta_1|||^2 + A_{vsc}(u_h;\zeta_1,\zeta_1) \;\le\; \mathcal{J}_1\Big\{\frac{16}{\theta}C(\Theta(\zeta_2))^2 + \frac{1}{12}\gamma^*|||\zeta_2|||^2 + \mathcal{J}_2\,|||u-u_h|||^2$$
$$+C\sum_{E\in\mathcal{T}_h}\big[\,\tau_E\,\beta_E^2 + \mathcal{D}_E\,\big]\,\big(\frac{h_E}{p}\big)^{2\ell}\,\|u\|^2_{\ell+1,E}\Big\}, \qquad (3.9.24)$$

where, $\mathcal{J}_1 := \dfrac{24\,\Psi_0}{(15\theta-2\gamma^*)\Psi_0 - 48(L_\Psi+\mu^*)} > 0$, and $\mathcal{J}_2 := \dfrac{96(L_\Psi+\mu^*) + 6\theta\Psi_0 + \gamma^*\Psi_0}{96\,\Psi_0}$.

Let us estimate $A_{vsc}(u_h;\zeta_2,\zeta_2)$. Using (3.7.11) and (3.8.30), we have

$$A_{vsc}(u_h;\zeta_2,\zeta_2) \;\le\; \gamma^*\,A_{sc}(u_h;\zeta_2,\zeta_2) \;\le\; \sum_{E\in\mathcal{T}_h}|\xi(u_h)|\zeta_2|^2_{1,E}$$
$$\le\; \sum_{E\in\mathcal{T}_h}\rho_E(u_h)\,|R_E^*(u_h)|^2\,|u-u_I|^2_{1,E}$$

Using (3.7.2) and (3.9.4), we get

$$A_{vsc}(u_h;\zeta_2,\zeta_2) \;\le\; \kappa\sum_{E\in\mathcal{T}_h}\tau_E\,\|-\nabla\cdot\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla u_h + \overrightarrow{\beta}\cdot\nabla u_h + \Psi(u_h) - f\|^2_{0,E}\,\frac{|u|^2_{1,E}}{||u_h|_{1,E}+\sigma_E|^2}$$
$$\le\; \kappa\,\mathcal{C}_u\sum_{E\in\mathcal{T}_h}\tau_E\,\|-\nabla\cdot\mathbf{\Pi}^0_{p-1}\mathcal{D}\nabla u_h + \overrightarrow{\beta}\cdot\nabla u_h + \Psi(u_h) - f\|^2_{0,E},$$

where, $\mathcal{C}_u := \big[\,\max_{E\in\mathcal{T}_h}\big\{\,|u|_{1,E}/||u_h|_{1,E}+\sigma_E|\,\big\}\,\big]^2$. Estimating similar to (3.9.17), noting that $\theta < 1$ and for sufficiently small $\kappa > 0$, having $\kappa\,\mathcal{C}_u \le \dfrac{\theta}{64}$, we obtain,

$$A_{vsc}(u_h;\zeta_2,\zeta_2) \le \frac{\gamma^*}{8}\Big(|||\zeta_1|||^2 + |||\zeta_2|||^2\Big) + C\sum_{E\in\mathcal{T}_h}\mathcal{D}_E\big(\frac{h_E}{p}\big)^{2\ell}\,\|u\|^2_{\ell+1,E} + \frac{\gamma^*}{64}|||u-u_h|||^2. \qquad (3.9.25)$$

Note that, $|||u-u_h|||^2 \le\; 2|||\zeta_1|||^2 + 2|||\zeta_2|||^2$ and $A_{vsc}(u_h;u-u_h,u-u_h) \le 2A_{vsc}(u_h;\zeta_1,\zeta_1) + 2A_{vsc}(u_h;\zeta_2,\zeta_2)$. Thus,

$$|||u-u_h|||^2 + A_{vsc}(u_h; u-u_h, u-u_h) \leq\, 2|||\zeta_1|||^2 + 2A_{vsc}(u_h;\zeta_1,\zeta_1) + 2|||\zeta_2|||^2 + 2A_{vsc}(u_h;\zeta_2,\zeta_2).$$

Therefore using (3.9.24), (3.9.25) and substituting (3.9.24) as a bound for $|||\zeta_1|||^2$, we get,

$$\begin{aligned}
&|||u-u_h|||^2 + A_{vsc}(u_h; u-u_h, u-u_h) \\
&\quad \leq \big(\, 2 + \frac{\gamma^*}{4}\,\big)\, \mathcal{J}_1 \Big\{\, \frac{16}{\theta} C(\Theta(\zeta_2))^2 + \frac{\gamma^*}{12}|||\zeta_2|||^2 + \mathcal{J}_2\, |||u-u_h|||^2 \,\Big\} \\
&\qquad + C \sum_{E\in\mathcal{T}_h} \big[\, \tau_E\, \beta_E^2 + \mathcal{D}_E \,\big]\, \big(\frac{h_E}{p}\big)^{2\ell}\, \|u\|^2_{\ell+1,E} + 2\,|||\zeta_2|||^2 + \frac{\gamma^*}{4}|||\zeta_2|||^2 + \frac{\gamma^*}{32}|||u-u_h|||^2.
\end{aligned}$$

Using $\theta < 1$, collecting the coefficients of $|||u-u_h|||^2$ and simplifying, we get

$$\begin{aligned}
\mathcal{J}_3\, |||u-u_h|||^2 \quad + \quad & A_{vsc}(u_h; u-u_h, u-u_h) \\
&\leq\, \big(\, 2 + \frac{\gamma^*}{4}\,\big)\, \mathcal{J}_1\, \frac{16}{\theta} C(\Theta(\zeta_2))^2 + \Big(\, 2 + \frac{\gamma^*}{4}\,\Big)\Big(\, \frac{\gamma^*}{12}\mathcal{J}_1 + 1\,\Big)\, |||\zeta_2|||^2 \\
&\quad + C \sum_{E\in\mathcal{T}_h} \big[\, \tau_E\, \beta_E^2 + \mathcal{D}_E \,\big]\, \big(\frac{h_E}{p}\big)^{2\ell}\, \|u\|^2_{\ell+1,E},
\end{aligned} \tag{3.9.26}$$

where, $\quad \mathcal{J}_3 := \dfrac{3[\,(8\theta-\gamma^*)\Psi_0 - (64+3\gamma^*)(L_\Psi+\mu^*)\,]}{2[\,(15\theta-2\gamma^*)\Psi_0 - 48(L_\Psi+\mu^*)\,]} > 0.$

Using (3.7.11) on $A_{vsc}(u_h; u-u_h, u-u_h)$ and absorbing the coefficients of $|||\zeta_2|||^2$ into the coefficient of $\Theta(\zeta_2)$ in (3.9.26), we obtain the desired estimate (3.9.10). □

3.9.2 Adding crosswind-direction diffusion.

Whenever $\overrightarrow{\beta} \neq \mathbf{0}$, the term $A_{vsc}(\cdot;\cdot,\cdot)$ adds artificial diffusion in a crosswind direction with the parameters P^{sc} and $\widehat{\xi}(\cdot)$ in (3.7.8) set as follows :

$$P^{sc} := \mathbf{I} - \frac{\overrightarrow{\beta} \otimes \overrightarrow{\beta}}{|\overrightarrow{\beta}|^2} \quad \text{and} \quad \widehat{\xi}_E(z) := \varrho_E(z)\, R^*_E(z), \tag{3.9.27}$$

where $R^*_E(z)$ is as defined in (3.7.9).

Theorem 3.5. *Let* $u \in H^1_0(\Omega) \cap H^{\ell+1}(E)$, $\ell \in \mathbb{N}$ *be the exact solution of* (3.6.1) *with* $(\nabla\cdot\mathcal{D}\nabla u)|_E \in L^2(E)$ *for all* $E \in \mathcal{T}_h$. *Let the stabilization parameter* τ_E *be as in* (3.9.9) *and* $\Psi_0\,(15\theta - 2\gamma^*) > 2(64+3\gamma^*)(L_\Psi+\mu^*)$ *be satisfied. For a sufficiently small* $\kappa > 0$, *we suppose that* $0 \,\leq\, \varrho_E(v_h) \,\leq\, \kappa\,\tau_E\, R^*_E(v_h) \quad \forall\, v_h \in V^p_h$. *Then the solution* $u_h \in V^p_h$ *of*

(3.7.13) *with* (3.9.27) *satisfies,*

$$
\begin{aligned}
&|||u-u_h|||^2+\gamma_*\sum_{E\in\mathcal{T}_h}\widehat{\xi}_E(u_h)\,\|(P^{sc})^{1/2}\,\nabla(u-u_h)\|^2_{0,E}\\
&\qquad\le\;C\Big\{\,[\Theta(u-u_I)]^2\;+\sum_{E\in\mathcal{T}_h}\big[\,\tau_E\,\beta_E^2+\mathcal{D}_E\,\big]\,\big(\frac{h_E}{p}\big)^{2\ell}\,\|u\|^2_{\ell+1,E}\Big\},\qquad(3.9.28)
\end{aligned}
$$

where $u_I \in V_h^p$ *is the virtual interpolant of* u *as in* (3.8.30) *and constant* $C>0$ *is dependent on* L_Φ, Ψ_0, θ, γ^*, *but independent of* $\mathcal{D}$, h_E.

Proof. The proof is similar to Theorem 3.4. □

Now we prove a convergence result with respect to the $|||\cdot|||$ for the discrete scheme (3.7.13) with either (3.9.4) or (3.9.27), using suitable choice for τ_E.

Theorem 3.6. *Let* $u \in H_0^1(\Omega)$ *be the solution of* (3.6.1) *with* $u\in H^{\ell+1}(E)$, $p\ge\ell>1$. *Let* $u_h \in V_h^p$ *satisfy problem* (3.7.13) *with one of the two variants,* (3.9.4) *or* (3.9.27). *Consider the assumptions on* Ψ_0, γ^* *and* $\widehat{\xi}_E(\cdot)$ *given in either Theorem 3.4 or Theorem 3.5, depending on the SC variants* (3.9.4) *or* (3.9.27)*, respectively. Additionally we assume* $\dfrac{1}{\tau_E}\le\dfrac{\beta_E^2}{\mathcal{D}_E^\vee}$ *and the choice for* τ_E *as,*

$$
\tau_E\sim\min\Big\{\frac{h_E}{p\beta_E}\;;\;\frac{h_E^2}{p^4\,c_{inv}^2\mathcal{D}_E}\;;\;\frac{\Psi_0}{L_\Psi^2}\Big\}.\qquad(3.9.29)
$$

Let us denote Peclet number $Pe_E:=\dfrac{h_E\,\beta_E}{p\,\mathcal{D}_E}$. *Then for sufficiently small* h, *we have*

$$
|||u-u_h|||^2\le C\sum_{E\in\mathcal{T}_h}\Big(\frac{h_E}{p}\Big)^{2\ell}\mathcal{D}_E\Big(1+Pe_E+\mathcal{Z}_E^{(t)}+\min\big\{\,\mathfrak{X}_E\,;\,\frac{\mathcal{D}_E}{\mathcal{D}_E^\vee}\,Pe_E^2\big\}\Big)\,\|u\|^2_{\ell+1,E},\quad(3.9.30)
$$

where, $\mathfrak{X}_E:=\max\Big\{Pe_E;\,p^2c_{inv}^2;\,\mathcal{Z}_E^{(r)}\Big\}$, $\mathcal{Z}_E^{(t)}:=\dfrac{\Psi_0}{\mathcal{D}_E}\Big(\dfrac{h_E}{p}\Big)^2$, $\mathcal{Z}_E^{(r)}:=\dfrac{L_\Psi^2}{\Psi_0\,\mathcal{D}_E}\Big(\dfrac{h_E}{p}\Big)^2$.

Proof. Using (3.9.10) or (3.9.28) and (3.8.22), we have,

$$
\begin{aligned}
|||u-u_h|||^2\;&\le\;C\,[\Theta(u-u_I)]^2+C\sum_{E\in\mathcal{T}_h}\big(\,\tau_E\,\beta_E^2+\mathcal{D}_E\,\big)\Big(\frac{h_E}{p}\Big)^{2\ell}\|u\|^2_{\ell+1,E}\\
&\le\;C\Big[|||u-u_I|||+\Big(\sum_{E\in\mathcal{T}_h}\min\Big\{\frac{1}{\tau_E};\frac{\beta_E^2}{\mathcal{D}_E^\vee}\Big\}\|u-u_I\|^2_{0,E}\Big)^{\frac12}\Big]^2\\
&\quad+C\sum_{E\in\mathcal{T}_h}\big(\,\tau_E\,\beta_E^2+\mathcal{D}_E\,\big)\Big(\frac{h_E}{p}\Big)^{2\ell}\|u\|^2_{\ell+1,E}
\end{aligned}
$$

Applying $(m+n)^2 \leq 2(m^2+n^2)$, the definition of $|||\cdot|||$ and (3.8.30), we get

$$
\begin{aligned}
|||u-u_h|||^2 \quad &\leq \quad C\Big[\sum_{E\in\mathcal{T}_h}\Big(\mathcal{D}_E|u-u_I|^2_{1,E}+\Psi_0\,\|u-u_I\|^2_{0,E}+\tau_E\,\beta_E^2|u-u_I|^2_{1,E}\Big)\\
&\quad + \sum_{E\in\mathcal{T}_h}\min\Big\{\frac{1}{\tau_E};\frac{\beta_E^2}{\mathcal{D}_E^{\vee}}\Big\}\|u-u_I\|^2_{0,E}+\sum_{E\in\mathcal{T}_h}\big(\,\tau_E\,\beta_E^2+\mathcal{D}_E\,\big)\Big(\frac{h_E}{p}\Big)^{2\ell}\|u\|^2_{\ell+1,E}\Big]\\
&\leq \quad C\sum_{E\in\mathcal{T}_h}\Big(2[\mathcal{D}_E+\tau_E\,\beta_E^2]+\frac{h_E^2}{p^2}\Psi_0+\frac{h_E^2}{p^2}\min\Big\{\frac{1}{\tau_E};\frac{\beta_E^2}{\mathcal{D}_E^{\vee}}\Big\}\Big)\Big(\frac{h_E}{p}\Big)^{2\ell}\|u\|^2_{\ell+1,E}.
\end{aligned}
$$

Note that, from (3.9.29), the definitions of Pe_E, $\mathcal{Z}_E^{(r)}$, we have $\dfrac{1}{\tau_E} = \mathfrak{X}_E$ and using $\tau_E \leq \dfrac{h_E}{p\beta_E}$ we obtain the desired estimate (3.9.30). $\square$

Let us now examine the optimality of (3.9.30) in the cases of convection dominated or reaction dominated phenomenon. For simplicity we assume $\mathcal{D}(x) \equiv \mathcal{D}$. Under the convection dominated case, ie. $Pe \geq \max\{\mathcal{Z}_E^{(r)}, \mathcal{Z}_E^{(t)}\} \geq p^2\,c_{\text{inv}}^2$, or the reaction dominated case, ie. $\min\{\mathcal{Z}_E^{(r)}, \mathcal{Z}_E^{(t)}\} \geq Pe \geq p^2\,c_{\text{inv}}^2$, from (3.9.30) we get,

$$
|||u-u_h|||^2 \leq C\sum_{E\in\mathcal{T}_h}\Big(\frac{h_E}{p}\Big)^{2s+1}\|u\|^2_{\ell+1,E}. \tag{3.9.31}
$$

Thus (3.9.31) implies optimal order of convergence in the $|||\cdot|||$.

3.10 Numerical Experiments

In this section we discuss two benchmark problems highlighting that shock capturing VEM reduces the oscillations along the layers more efficiently than VEM-SUPG method. In our simulations we have considered VEM of orders $p = 1,\ 2$ and 3. For computational purpose, we choose the stabilization parameter $\tau_E := \min\Big\{\dfrac{h_E}{2\,\|\overrightarrow{\beta}\|_{\mathbb{R}^2}},\ \dfrac{h_E^2}{|\mathcal{D}|}\Big\}$ and for $\varrho_E(\cdot)$ in (3.9.27), we consider (see [69]) :

$$
\varrho_E(z) := q_0\,h_E\,\max\Big\{\,0,\ \delta-\frac{2\,|\mathcal{D}|}{h_E\,R_E^*(z)}\,\Big\}, \tag{3.10.1}
$$

with $q_0 \in [0.1, 1], \delta = 0.7$ and $\sigma = 10^{-4}$ in (3.7.9).

For both the examples, we consider $\Omega = (0,1)^2$. The discretized nonlinear system of equations were solved using Newton's method with zero initial guess. The stopping criteria for the iteration is fixed as 10^{-7}. The error with respect to the energy norm $|||\cdot|||$ is denoted by e_h and defined as,

$$e_h^2 \;=\; \sum_{E\in\mathcal{T}_h} \Big(\|\sqrt{\mathcal{D}}\nabla(u-\Pi_p^{\nabla}u_h)\|_{0,E}^2 + \Psi_0\,\|u-\Pi_p^0 u_h\|_{0,E}^2 + \tau_E\,\|\overrightarrow{\beta}\cdot\nabla(u-\Pi_p^{\nabla}u_h)\|_{0,E}^2 \Big)\,.$$

3.10.1 Example 1

We consider $\mathcal{D}=10^{-7}$, $\overrightarrow{\beta}=(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})^T$ and $\Psi(u)=u+u^3$ in (3.6.1). The source function f is defined by choosing the exact solution as $u(x,y):=\frac{1}{2}\Big(1-\tanh\frac{2x_1-x_2-0.25}{\sqrt{5\mathcal{D}}}\Big)$. We use Dirichlet boundary conditions prescribed by u. Note that the solution is dependent on the diffusion coefficient $\mathcal{D}$ and is characterised by an interior layer of $O(\sqrt{\mathcal{D}}|ln\,\mathcal{D}|)$ around the line $2x_1-x_2-0.25$.

We consider regular Voronoi mesh (Fig. 3.8) whose important parameters are presented in Table 3.2. We compute the numerical solution of this problem using the discrete scheme (3.7.13) with artificial crosswind-direction diffusion terms given in (3.9.27). Let us take $q_0=0.1$ in (3.10.1).

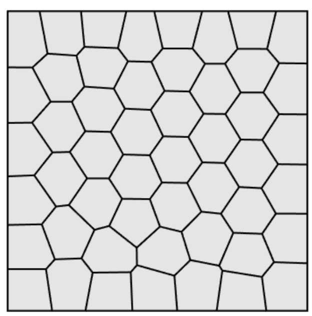


Figure 3.8: Sample of regular Voronoi mesh with h=1/5.

h	N_E	dof p=1	dof p=2	dof p=3
1/5	80	162	483	884
1/10	300	601	1801	3301
1/20	1300	2599	7797	14295
1/40	5000	9998	29995	54992
1/80	24000	47959	143917	263875

Table 3.2: Regular Voronoi mesh parameters with mesh diameter (h), number of elements (N_E) and degrees of freedom (dof) for VEM orders p=1,2 and 3.

The errors e_h obtained for the regular Voronoi mesh for VEM orders $p=1,2$ and 3, along with the rate of convergence is given in Table 3.3. Since the solution u is dependent on $\mathcal{D}$, the optimal order of convergence will be obtained only for very small h.

	p=1		p=2		p=3	
h	e_h	roc	e_h	roc	e_h	roc
1/5	$7.36e^{-2}$	$*$	$9.24e^{-2}$	$*$	$7.38e^{-2}$	$*$
1/10	$6.31e^{-2}$	0.22	$8.26e^{-2}$	0.16	$4.95e^{-2}$	0.57
1/20	$4.47e^{-2}$	0.49	$6.19e^{-2}$	0.41	$3.35e^{-2}$	0.56
1/40	$3.68e^{-2}$	0.28	$4.35e^{-2}$	0.51	$2.91e^{-2}$	0.21
1/80	$2.53e^{-2}$	0.54	$2.66e^{-2}$	0.71	$1.57e^{-2}$	0.89

Table 3.3: Error e_h wrt $|||\cdot|||$ and the rate of convergence (roc).

In order to show the effect of adding shock capturing stabilization term in the formulation, we compare the cross-sectional graph of the SUPG stabilized VEM method with and without shock capturing term. In Figure 3.9-3.10, we consider the cross-section along the line $x_1 + 2x_2 = 1$.

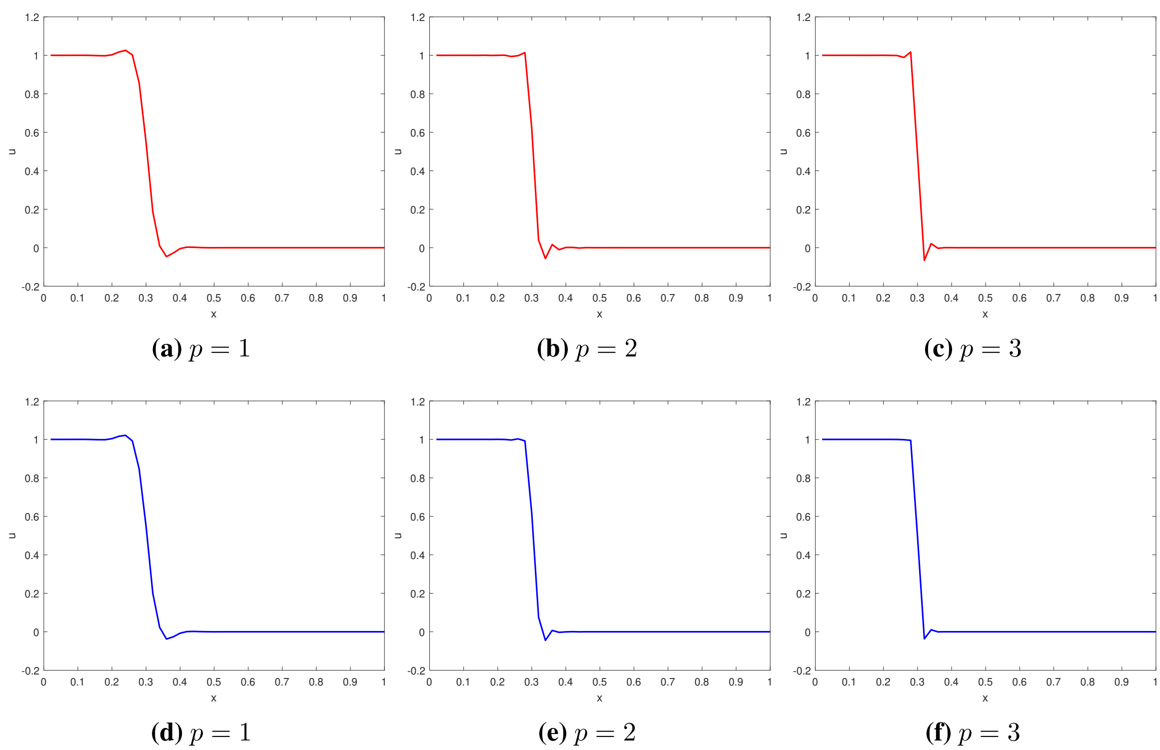
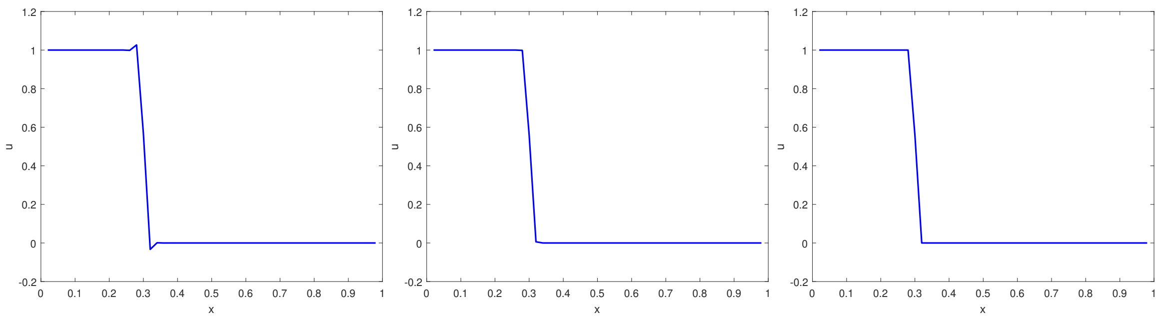


Figure 3.9: Cross-section graph : VEM-SUPG (top) and VEM-SUPG+SC (bottom) for regular Voronoi mesh with h=1/20.



(a) $p = 1$ **(b)** $p = 2$ **(c)** $p = 3$

Figure 3.10: Cross-section graph : VEM-SUPG+SC for regular Voronoi mesh with h=1/80.

From Figure 3.9 we infer that the oscillations are effectively damped effectively by the Shock capturing VEM of order greater than one. We can also observe that the quality of the numerical solution increase with increase in VEM order p. In Figure 3.10 we see that for $h = 1/80$, the oscillations are completely removed for VEM orders $p = 2$ and 3.

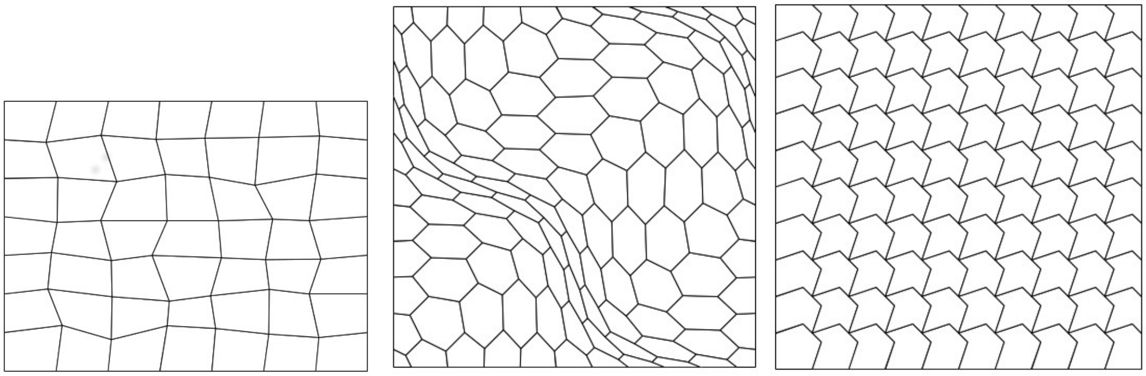
3.10.2 Example 2

We consider $\mathcal{D} = 10^{-6}$, $\overrightarrow{\beta}(x, y) = (-y, x)^T$, $\Psi(u) = u^4$ and $f = 0$. The discontinuous boundary data is prescribed as follows :

$$\begin{cases} u(x,y) = 1 & \text{if } 1/3 \leq x \leq 2/3,\ y = 0 \\ u(x,y) = 0 & \text{if } x \in [0, 1/3) \cap (2/3, 1],\ y = 0 \\ u(x,y) = 0 & \text{if } x = 1,\ y \in [0,1] \\ u(x,y) = 0 & \text{if } x \in [0,1],\ y = 1 \\ \dfrac{\partial u(x,y)}{\partial \mathbf{n}} = 0 & \text{if } x = 0,\ y \in [0,1], \end{cases} \tag{3.10.2}$$

where $\mathbf{n}$ is the unit outward normal. We use the discrete scheme (3.7.13) with artificial crosswind-direction diffusion terms given in (3.9.27) and $q_0 = 0.2$ in (3.10.1) to compute the numerical solution. The solution u possess two interior characteristic layers beginning from the line joining the points (1/3,0) and (2/3,0).

We consider three different meshes namely, distorted squares, hexagons and non-convex polygons. A representative of each mesh is shown in Figure 3.11. In Table 3.4 we present details of some useful mesh parameters.



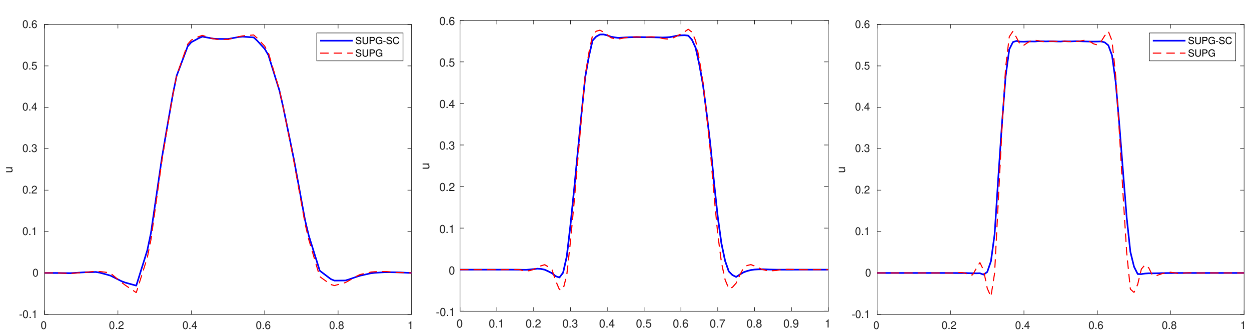
(a) Distorted squares. **(b)** Hexagons. **(c)** Non-convex polygons.

Figure 3.11: Samples of meshes with diameter $h = 1/5$.

Mesh	h	N_E	dof p=1	dof p=2	dof p=3
Distorted	1/16	784	841	3249	6441
squares	1/32	3136	3249	12769	25425
Hexagons	1/16	1681	3364	10089	18495
	1/32	6561	13124	39369	72175
Non-convex	1/16	1600	4801	12801	22401
polygons	1/32	6400	19201	51201	89601

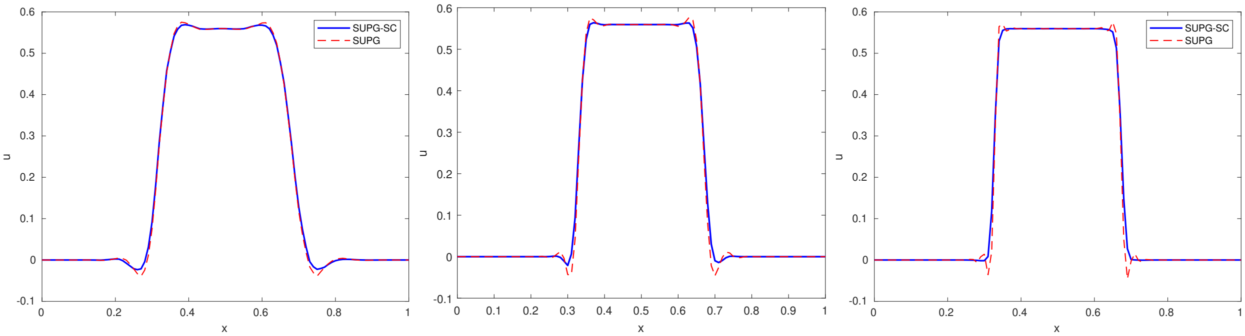
Table 3.4: Mesh parameters with degrees of freedom (dof) and number of elements (N_E).

We study the performance of our shock capturing VEM method by comparing with SUPG stablised VEM on the cross-section of the outflow boundary line $x = 0$. In Figure 3.12-3.13 presents the outflow boundary cross-section of the numerical solution for distorted square mesh with $h = 1/16$ and $h = 1/32$, respectively, for VEM order $p = 1, 2, 3$.



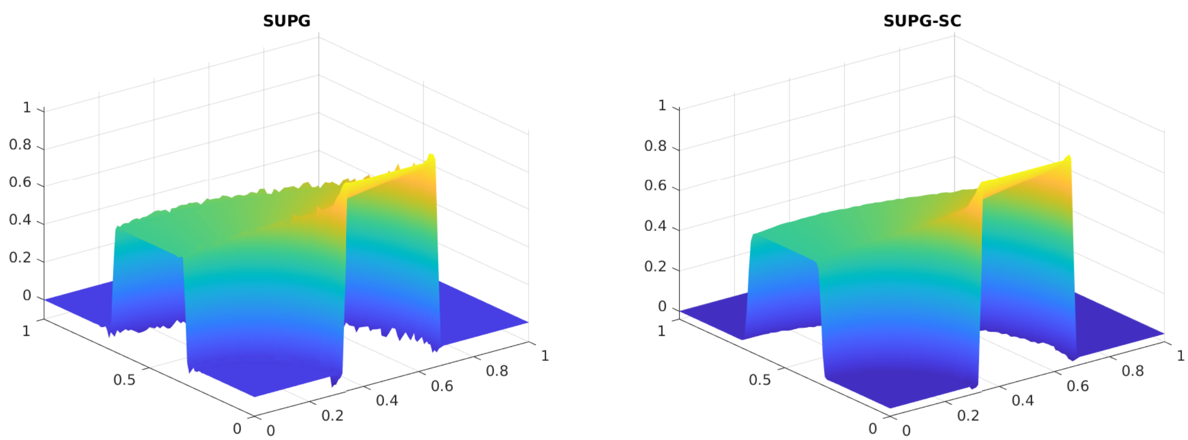
(a) $p = 1$ **(b)** $p = 2$ **(c)** $p = 3$

Figure 3.12: Cross-section graph : VEM-SUPG (red broken line) VEM-SUPG+SC (blue line) for distorted square mesh with h=1/16.



(a) $p = 1$ **(b)** $p = 2$ **(c)** $p = 3$

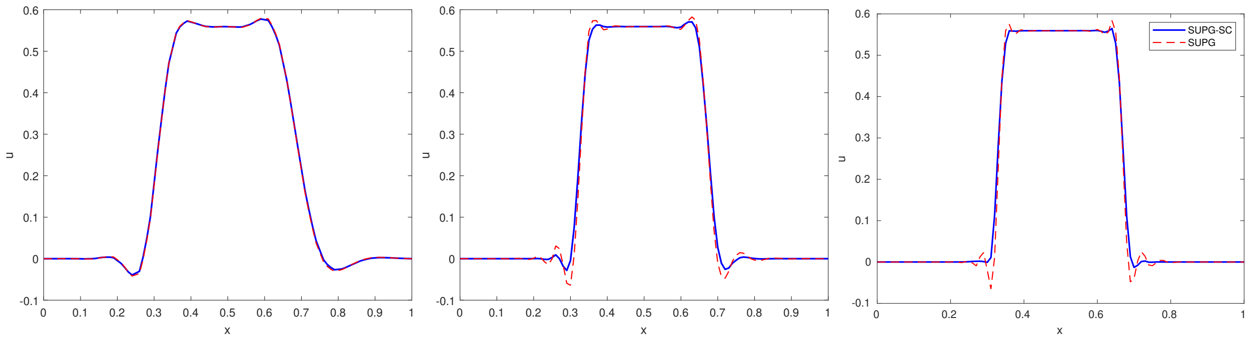
Figure 3.13: Cross-section graph : VEM-SUPG (red broken line) VEM-SUPG+SC (blue line) for distorted square mesh with h=1/32.



(a) $p = 3$ **(b)** $p = 3$

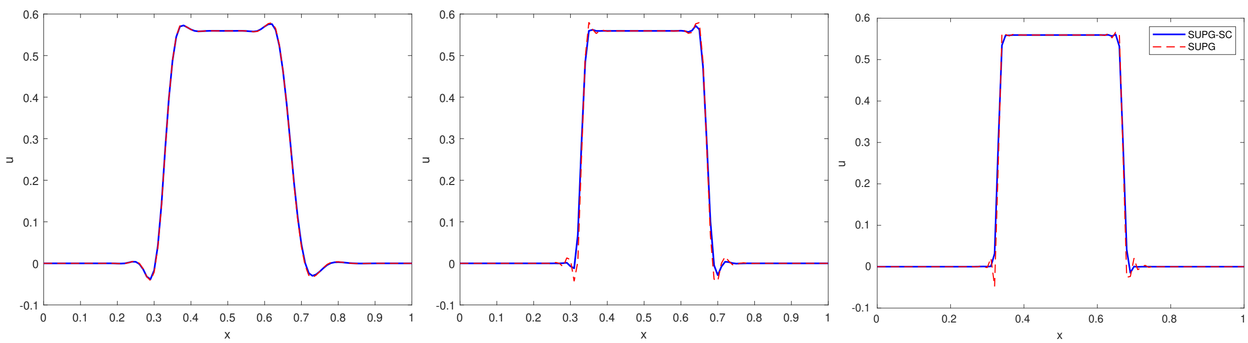
Figure 3.14: A comparison of surface plot of numerical solution obtained without- and with- shock cpaturing for distoted square mesh with $h = 1/32$ and VEM order p=3.

Similarly, Figure 3.15-3.16 represents the outflow boundary cross-section of the numerical solution for hexagonal mesh with $h = 1/16$ and $h = 1/32$, respectively, for VEM order $p = 1, 2$ and 3. Again, Figure 3.17-3.18 represents the outflow boundary cross-section of the numerical solution for non-convex mesh with $h = 1/16$ and $h = 1/32$, respectively, for VEM order $p = 1, 2$ and 3.



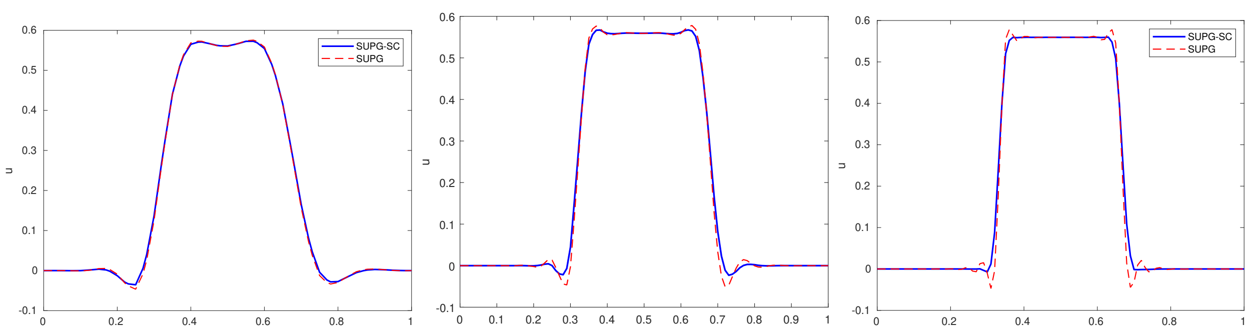
(a) $p = 1$ **(b)** $p = 2$ **(c)** $p = 3$

Figure 3.15: Cross-section graph : VEM-SUPG (red broken line) VEM-SUPG+SC (blue line) for hexagonal mesh with h=1/16.



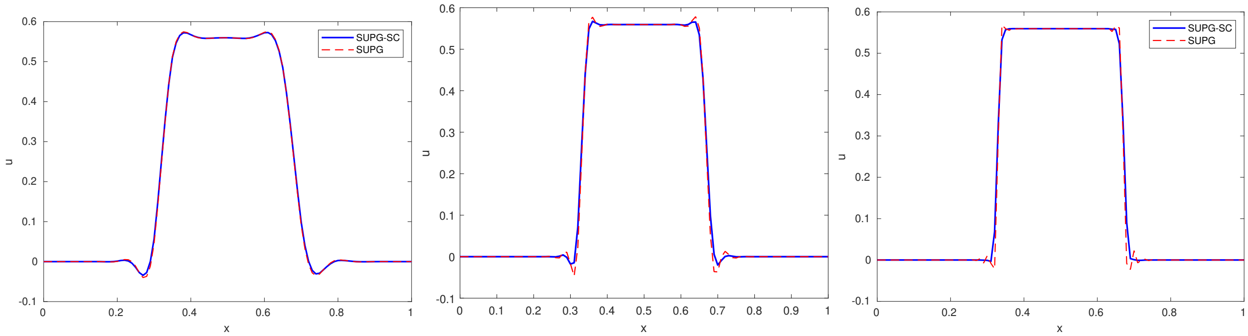
(a) $p = 1$ **(b)** $p = 2$ **(c)** $p = 3$

Figure 3.16: Cross-section graph : VEM-SUPG (red broken line) VEM-SUPG+SC (blue line) for hexagonal mesh with h=1/32.



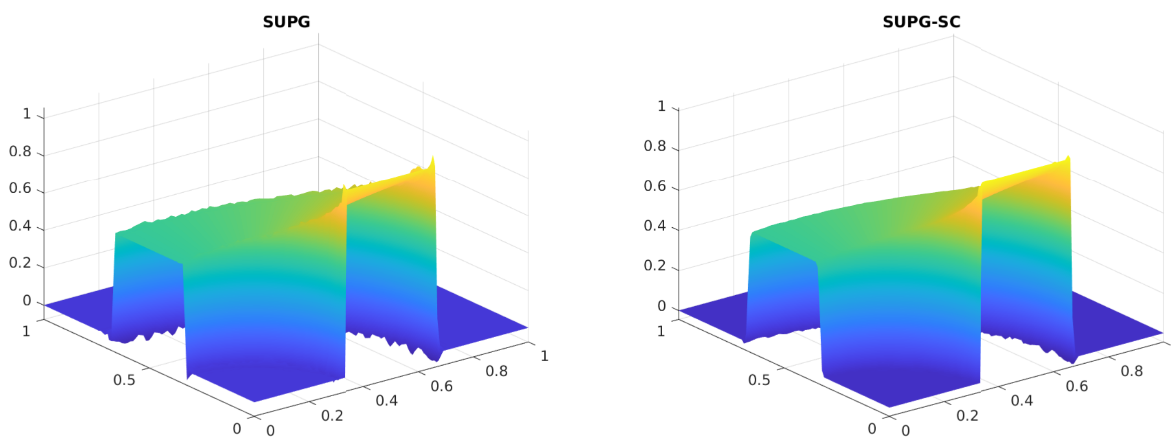
(a) $p = 1$ **(b)** $p = 2$ **(c)** $p = 3$

Figure 3.17: Cross-section graph : VEM-SUPG (red broken line) VEM-SUPG+SC (blue line) for non-convex mesh with h=1/16.



(a) $p = 1$ **(b)** $p = 2$ **(c)** $p = 3$

Figure 3.18: Cross-section graph : VEM-SUPG (red broken line) VEM-SUPG+SC (blue line) for non-convex mesh with h=1/32.

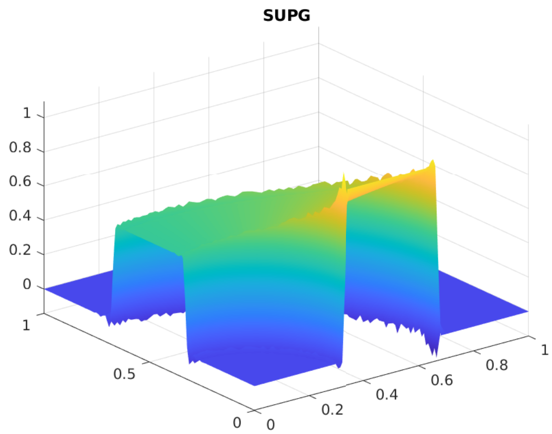


(a) $p = 3$ **(b)** $p = 3$

Figure 3.19: A comparison of surface plot of numerical solution obtained without- and with- shock cpaturing for hexagonl mesh with $h = 1/32$ and VEM order p=3.

In Figures 3.14, 3.19 and 3.20, we present the surface plot of the numerical solution obtained from SUPG stabilized VEM, and shock capturing VEM of order $p = 3$ for different meshes such as distorted squares, hexagons and non-convex polygons, respectively, for $h = 1/32$. Clearly, we see the efficiency of the shock-capturing term in reducing the nonphysical oscillations in the numerical solution on the three meshes considered.

Across all meshes taken into consideration, from Figures 3.12-3.13, 3.15-3.16, and 3.17-3.18, we infer the following about the effectiveness of shock-capturing stabilization term in (3.7.13). In the case of linear VEM, there is negligible reduction of spurious oscillations in the numerical solution. But in higher order VEM, the shock-capturing term effectively diminishes the nonphysical oscillation of the numerical solution in the layer regions. Also, in a particular higher order VEM, reducing the diameter h produces highly qualitative numerical solution.



(a) $p = 3$

(b) $p = 3$

Figure 3.20: A comparison of surface plot of numerical solution obtained without- and with- shock capturing for non-convex mesh with $h = 1/32$ and VEM order p=3.

3.11 Summary

This chapter has studied the shock-capturing stabilized VEM for the convection-diffusion-reaction equation. As a motivation, we formulated a well-posed shock capturing stabilized discrete scheme approximating the linear convection-diffusion-reaction equation in the VEM context. Numerical experiments conducted on linear problems with unknown solutions having discontinuous boundary data revealed the efficiency of shock-capturing technique over the SUPG method on second-order VEM for different types of meshes. Since the exact solution was not known, a comparison of the cross-section of the corresponding first-order FEM and VEM solution on triangular mesh exhibit similar structures. We have devised a shock-capturing stabilization of the VEM for a semilinear convection-diffusion-reaction equation with this boosting. An extensive theoretical analysis of the approximate scheme was conducted. We have shown the well-posedness of the formulation and its error estimates with the convergence rate. We laid the conditions for choosing optimal SUPG parameters. In the end, we performed two numerical experiments to validate the theoretical findings. Both the experiments show the effectiveness of the shock-capturing VEM compared with the VEM-SUPG method.

For first-order VEM, the shock-capturing method was ineffective for the linear and nonlinear problems. We highlight that the shock-capturing technique combined with higher-order VEM efficiently damps the spurious oscillations in the numerical solution.

Chapter 4

Virtual element method for the quasilinear convection-diffusion-reaction equation on polygonal meshes

The quasilinear convection-diffusion equation arises in diverse areas such as in plasma physics describing movement of ions [71], the Burgers equation related to turbulence theory [72], gas and oil extractions, fibre optics and aerodynamic theory [73]. In order to reduce the spurious oscillations appearing in the numerical solution of convection dominated problem, we use the streamline upwind Petrov-Galerkin method for stabilizing the virtual element method. We know that quasilinear convection-diffusion equations are closely related to the Navier-Strokes equation. In many practical applications, the solution of these equations are isolated, that is, the solution is unique upto a neighbourhood. We call this collection of solutions in a neighbourhood as a branch of solutions. This chapter studies the VEM approximation of branch of nonsingular solution of quasilinear convection-diffusion-reaction equation. The analysis is based on a variant of broader theory developed for a class of nonlinear problems by Brezzi et. al. [74].

The main feature of virtual element space is that the associated local degrees of freedom uniquely determines the functions in the interior and on the boundary of each element. Since the non-polynomial component of the functions are not known explicitly, neither we have an approximate expression for the basis function nor we can use quadrature formula to compute the discrete scheme. Thus, we must take special care in devising the discrete operators in the VEM scheme. In the VEM context, we use polynomial projection operators on the functions to split it into its polynomial and non-polynomial constituents, and the operator terms are evaluated using only the degrees of freedom such that we obtain exact results when one of the two entries is a polynomial, and for other occurrences we produce values of only right order of magnitude and stability property. Hence to ensure computablity, we use projection operators appropriately in the discrete formulation. Moreover, to approximate the nonlinear convective coefficient and reaction function, we incorporate the projection operators and add the necessary VEM stabilizers. From the analysis, we note that for the VEM stabilizers supporting the nonlinear reaction function, it is sufficient to provide a linear coefficient, whereas in the approximation involving nonlinear convective function, the coefficient of the VEM stabilizer remained nonlinear. In the error analysis section, we show that the use of polynomial projection operators and the added VEM stabilizers do not affect the rate of convergence.

A challenge in numerical simulation of quasilinear problem is the execution of Newton's iterative method which becomes computationally expensive on very fine mesh. In two-grid method, we solve the system with two meshes of different mesh diameters. The nonlinear system is solved on a much coarser grid. Then, in the fine grid, only a few number (say, one or two) of nonlinear iterations are performed with the coarse grid solution as the initial guess. Various adaptations of two-grid methods have been successfully applied to many problems such as quasilinear elliptic equation [75], nonlinear hyperbolic equation [76], nonlinear parabolic integro-differential equations [77] and mixed FEM for Darcy-Forchheimer model [78]. Hence in the numerical simulations, we consider using the two-grid method proposed in [79] for solving the discrete formulation.

4.1 The continuous problem

Consider the model problem,

$$\begin{aligned} \mathcal{L}(u) := -\nabla \cdot (\epsilon \nabla u) + \boldsymbol{\beta}(u) \cdot \nabla u + r(u) &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{4.1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary Γ. We assume the parameter $\epsilon \in \mathbb{R}^+$ where $0 < \epsilon_0 \leq \epsilon \leq \epsilon_1$,, the nonlinear coefficients $\boldsymbol{\beta}(\cdot) = (\beta_1(\cdot), \beta_2(\cdot))$ with $\beta_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2$ and $r : \mathbb{R} \to \mathbb{R}$ are twice differentiable functions. Let $\partial_u(\cdot)$ and $\partial_{uu}(\cdot)$ denote the first and second order derivatives with respect to u, respectively.

Assumption 4.1. Furthermore, we assume, the nonlinear function r such that $r(0) = 0$, and there exists a monotonically increasing function $\mathcal{Q} : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\sum_{j=1,2} \big[|\beta_j(s)| + |\partial_u \beta_j(s)| + |\partial_{uu} \beta_j(s)|\big] + |\mathrm{div}\, \boldsymbol{\beta}(s)| \\ +|r(s)| + |\partial_u r(s)| + |\partial_{uu} r(s)| \leq \mathcal{Q}(|s|) \qquad \forall s \in \mathbb{R}. \tag{4.1.2}$$

4.1.1 Notation

Let $\omega \subset \mathbb{R}^2$ be a measurable set. The usual Lebesgue space $L^2(\omega)$ is endowed with L^2 inner product denoted by $(\cdot,\cdot)_\omega$ and norm by $\|\cdot\|_{0,\omega}$, respectively. L^∞ norm denoted by $\|\cdot\|_{\infty,\omega}$. For the Sobolev space, $H^s(\omega), s \in \mathbb{N}$, we denote the seminorm by $|\cdot|_{s,\omega}$ and norm by $\|\cdot\|_{s,\omega}$. $\langle\cdot,\cdot\rangle$ denotes the duality pairing between a Banach space X_1 and its corresponding dual space X_2. Let $\mathrm{L}(X_2, X_1)$ be the space of bounded linear operator from X_2 into X_1, with standard operator norm $\|\cdot\|_{L(X_2,X_1)}$. We omit the index ω whenever the domain is evident.

4.1.2 The variational formulation of (4.1.1)

Find $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\langle \mathcal{L}(u),\, v\rangle \,:=\, \epsilon(\nabla u, \nabla v)_\Omega + (\boldsymbol{\beta}(u)\cdot\nabla u, v)_\Omega + (r(u), v)_\Omega = 0 \qquad \forall v \in H_0^1(\Omega). \tag{4.1.3}$$

We reformulate (4.1.3) to align with the abstract framework of Brezzi et.al [74, 80]. Let us denote $\mu_1 = \epsilon_1^{-1}$, $\mu_2 = \epsilon_0^{-1}$ and consider the compact interval $I = [\mu_1,\, \mu_2] \subset \mathbb{R}$. Then, for any $\epsilon \in [\epsilon_0,\, \epsilon_1]$, we have $\epsilon^{-1} \in I$. We know that [81], there exists an operator $M_0 : I \times H_0^1(\Omega) \to H^{-1}(\Omega)$ such that [81]:

$$\langle M_0(\mu, w),\, v\rangle := \,\mu\big[(\boldsymbol{\beta}(w)\cdot\nabla w, v)_\Omega + (r(w), v)_\Omega\big] \quad \forall v \in H_0^1(\Omega), \tag{4.1.4}$$

and a bounded linear inverse Laplace operator $\mathrm{T} : H^{-1}(\Omega) \to H_0^1(\Omega)$ solving

$$(\nabla(\mathrm{T}(g)), \nabla v)_\Omega = \langle g,\, v\rangle \quad \forall v \in H_0^1(\Omega). \tag{4.1.5}$$

Let $\mu = \epsilon^{-1}$. Using the operators in (4.1.4) and (4.1.5), the variational form (4.1.3) is rewritten as

$$\begin{cases} \text{find } u \in H_0^1(\Omega) \cap L^\infty(\Omega), \text{ such that} \\ F(\mu, u) := u + \mathrm{T}M_0(\mu, u) = 0. \end{cases} \tag{4.1.6}$$

In addition to the supposition (4.1.2), we assume the following (see [81]).

Assumption 4.2. There exists a branch $\mathcal{B} := \big\{(\mu, u_\mu) : \mu \in \mathrm{I},\ u_\mu \in H_0^1(\Omega) \cap L^\infty(\Omega) \big\}$ of nonsingular solutions of (4.1.6), in the sense that :

$$\text{for all } \mu \in I,\ \ F(\mu,\, u_\mu) = 0,$$

$$\text{the function } U : I \to H_0^1(\Omega) \text{ s.t. } U(\mu) = u_\mu \text{ is continuous , and}$$

$$\text{for all } \mu \in I,\ \ D\,F(\mu,\, u_\mu) \text{ is an isomorphism on } H_0^1(\Omega) \cap L^\infty(\Omega),$$

where $D\,F(\cdot, z)$ is the Fr$\acute{e}$chet derivative of operator F with respect to z.

Assumption 4.3. The set containing H^1 and L^∞ norms of solutions u_μ in branch $\mathcal{B}$ is uniformly bounded, i.e., $\exists\ \lambda > 0$ such that $\max\limits_{(\mu,\, u_\mu) \in \mathcal{B}} \{\, \|u_\mu\|_{1,\Omega}\,;\ \|u_\mu\|_{\infty,\Omega} \,\} \,\leq\, \lambda.$

Assumption 4.4. The map $\varphi :\ W^{2,p}(\Omega) \cap H_0^1(\Omega) \to L^p(\Omega)$ defined by $\varphi(w) = \Delta w$ is an isomorphism for all $p \in [1, 2]$.

Then (4.1.2), assumptions 4.2 and 4.4 implies any solution u satisfying the problem (4.1.6) belongs to $H^2(\Omega) \cap H_0^1(\Omega) \cap L^\infty(\Omega)$ (see *Lemma 2.1* in [81]).

Remark 4.1. Let $\bar{X} = H_0^1(\Omega) \cap H^2(\Omega)$ and $Y = L^2(\Omega)$. Using assumption 4.4, we also note $\mathrm{T} \in \mathrm{L}(Y, \bar{X})$.

4.1.3 VEM Spaces

Consider $\{\mathcal{T}_h\}_{h>0}$ to be a family of polygonal partitioning of Ω satisfying the assumption 1.1 stated in Chapter 1. In our analysis, we use the polynomial projection operators Π_p^∇, Π_p^0 and $\boldsymbol{\Pi}_p^0$ defined in (1.3.1), (1.3.2) and (1.3.3), respectively. For approximation we consider the global virtual element space V_h^p given in (1.3.5).

4.2 VEM formulation

It is a well known fact that the problem (4.1.1) is singularly perturbed, and the discretisation of (4.1.3) yields numerical solutions with non-physical oscillations. To alleviate this, we add the streamline-upwind Petrov-Galerkin (SUPG) stabilization in the discrete formulation. The SUPG stabilized discrete formulation is

$$\begin{cases} \text{find } u_h \in V_h^p \text{ such that} \\ a(u_h, v_h) + b(\{\, u_h, u_h \,\}; u_h, v_h) + c(u_h; u_h, v_h) + d(u_h; u_h, v_h) = 0 \quad \forall v \in V_h^p(\Omega) \end{cases}$$

where,

$$a(u_h, v_h) := \sum_{E\in\mathcal{T}_h} \epsilon\,(\,\nabla u_h,\,\nabla v_h\,)_E\,, \tag{4.2.1}$$

$$b(\{\,w_h, z_h\,\}; u_h, v_h) := \sum_{E\in\mathcal{T}_h} (\,\boldsymbol{\beta}(w_h)\cdot\nabla u_h,\,\delta_E\,\boldsymbol{\beta}(z_h)\cdot\nabla v_h\,)_E\,, \tag{4.2.2}$$

$$c(w_h; u_h, v_h) := \sum_{E\in\mathcal{T}_h} (\,r(u_h) + \boldsymbol{\beta}(w_h)\cdot\nabla u_h,\,v_h\,)_E\,, \qquad \text{and} \tag{4.2.3}$$

$$d(z_h; u_h, v_h) := \sum_{E\in\mathcal{T}_h} \left(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla u_h + r(u_h),\,\delta_E\,\boldsymbol{\beta}(z_h)\cdot\nabla v_h\,\right)_E. \tag{4.2.4}$$

The variable δ_E is the local stabilization parameter usually dependent on h_E. The functions in the VEM space V_h^p are only implicitly known through their degrees of freedom. Hence we need to suitably modify the terms in (4.2.1)-(4.2.4) so that they are computable using only the degrees of freedom.

To this end, we consider a symmetric bilinear form $S_E : V_h^E \times V_h^E \to \mathbb{R}$ such that, $\exists$ constants $\alpha_*,\,\alpha^* > 0$ independent of h and E satisfying

$$\alpha_*\,(\nabla v_h, \nabla v_h)_E \le S^E(v_h, v_h) \le \alpha^*\,(\nabla v_h, \nabla v_h)_E \qquad \forall v_h \in \ker \Pi_p^{\nabla}. \tag{4.2.5}$$

Using the polynomial projection operators Π_k^{∇}, Π_p^0, $\mathbf{\Pi}^0_{p-1}$, and S_E, we define the VEM computable terms as follows :

$$a_h(u_h, v_h) := \sum_{E\in\mathcal{T}_h} (\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla u_h,\ \mathbf{\Pi}^0_{p-1}\nabla v_h)_E + \epsilon\,S_E((I-\Pi_k^{\nabla})u_h, (I-\Pi_k^{\nabla})v_h), \tag{4.2.6}$$

$$\begin{aligned} b_h(\{w_h, z_h\}\,; u_h, v_h) := \sum_{E\in\mathcal{T}_h} \Big[& \left(\boldsymbol{\beta}(\Pi_p^0 w_h)\cdot\mathbf{\Pi}^0_{p-1}\nabla u_h,\ \delta_E\,\boldsymbol{\beta}(\Pi_p^0 z_h)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\right)_E \\ & + \delta_E\,\tilde{\mathrm{S}}^2\,S_E((I-\Pi_k^{\nabla})u_h, (I-\Pi_k^{\nabla})v_h)\Big], \end{aligned} \tag{4.2.7}$$

$$c_h(w_h; u_h, v_h) := \sum_{E\in\mathcal{T}_h} \left(r(\Pi_p^0 u_h) + \boldsymbol{\beta}(\Pi_p^0 w_h)\cdot\mathbf{\Pi}^0_{p-1}\nabla u_h,\ \Pi_p^0 v_h\right)_E, \tag{4.2.8}$$

$$d_h(z_h; u_h, v_h) := \sum_{E\in\mathcal{T}_h} \left(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla u_h + r(\Pi_p^0 u_h),\ \delta_E\,\boldsymbol{\beta}(\Pi_p^0 z_h)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\right)_E. \tag{4.2.9}$$

The parameter $\tilde{\mathrm{S}}$ in (4.2.7) is chosen guaranteeing two positive constants $\wp_*,\,\wp^*$ independent of h and E such that for all $w_h, z_h, v_h \in V_h^p$,

$$\wp_*\,b(\{w_h, z_h\}\,; v_h, v_h) \le b_h(\{w_h, z_h\}\,; v_h, v_h) \le \wp^*\,b(\{w_h, z_h\}\,; v_h, v_h), \tag{4.2.10}$$

and $\qquad \tilde{\mathrm{S}} \le \mathcal{Q}(\lambda).$ (4.2.11)

Similarly, for sufficiently small constants $C_1,\,C_2$ (independent of $\epsilon,\,h$) and for each $E \in$

$\mathcal{T}_h$, the choice for δ_E satisfies,

$$(i) \quad 0 \le \epsilon\, \delta_E \le C_1\, h_E^2, \qquad (ii) \quad \delta_E \le C_2\, h_E. \tag{4.2.12}$$

Let us denote,

$$\mathcal{A}(\{w_h, z_h\}\,; u_h, v_h) := a_h(u_h, v_h) + b_h(\{w_h, z_h\}\,; u_h, v_h) + c_h(w_h; u_h, v_h) + d_h(z_h; u_h, v_h).$$

A computable SUPG stabilized virtual element discretisation (VEM-SUPG) of (4.1.1) is,

$$\begin{cases} \text{find } u_h \in V_h^p \text{ such that} \\ \mathcal{A}(\{u_h, u_h\}\,; u_h, v_h) = 0 \qquad \forall v_h \in V_h^p. \end{cases} \tag{4.2.13}$$

Next, we re-write (4.2.13) as a discrete approximation to (4.1.6) and validate the existence of a branch of discrete solution which approximates the branch $\mathcal{B}$ of non-singular solutions given in assumption 4.2.

In the sequel, we denote by C a generic positive constant independent of $h_E,\ h,\ k$ and μ, which takes different values at different instances.

Lemma 4.1. *For $\phi_h,\ w_h \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $v_h \in V_h^k$, we define*

$$\sigma_h(\phi_h; w_h, v_h) := \mu\big[\, b_h(\{w_h, \phi_h\}\,; w_h, v_h) + c_h(w_h; w_h, v_h) + d_h(\phi_h; w_h, v_h)\,\big].$$

Under the conditions (4.1.2)*, assumption 4.2 and* (4.2.12)*, we have that $\sigma_h(\phi_h; w_h, \cdot)$ is a bounded linear functional on V_h^p.*

Proof. Given $\phi_h,\ w_h \in L^\infty(\Omega)$ implies that $\exists\ N > 0$ such that $\|\phi_h\|_{\infty,\Omega},\ \|w_h\|_{\infty,\Omega} \le N$. Let $K_1 := |\, b_h(\{w_h, \phi_h\}\,; w_h, v_h)\,|$. Using the conditions stated in lemma, (4.2.11) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} K_1 \;&\le\; (\mathcal{Q}(N))^2 \sum_{E\in\mathcal{T}_h} \big[\, \delta_E\, \|\mathbf{\Pi}_{p-1}^0 \nabla w_h\|_E\, \|\mathbf{\Pi}_{p-1}^0 \nabla v_h\|_E \\ &\qquad\qquad + \delta_E\, \alpha_*\, \|(I - \mathbf{\Pi}_{p-1}^0)\nabla w_h\|_E\, \|(I - \mathbf{\Pi}_{p-1}^0)\nabla v_h\|_E\,\big] \\ &\le\; C\, h\, (1+\alpha_*) \sum_{E\in\mathcal{T}_h} \|\nabla w_h\|_E\, \|\nabla v_h\|_E \qquad \text{(use (4.2.12))} \\ &\le\; C\, h\, (1+\alpha_*) \Big(\sum_{E\in\mathcal{T}_h} \|\nabla w_h\|_E^2\Big)^{\frac{1}{2}} \Big(\sum_{E\in\mathcal{T}_h} \|\nabla v_h\|_E^2\Big)^{\frac{1}{2}} \qquad \text{(using Hölder's inequality)} \\ &\le\; C\, h\, (1+\alpha_*)\, |w_h|_{1,\Omega}\, |v_h|_{1,\Omega}. \end{aligned} \tag{4.2.14}$$

Let $K_2 := |\, c_h(w_h; w_h, v_h)\,|$. Similarly, we have

$$\begin{aligned}
K_2 &\leq C \sum_{E\in\mathcal{T}_h} \mathcal{Q}(N) \left[\, \|\Pi^0_p v_h\|_E + \|\mathbf{\Pi}^0_{p-1}\nabla w_h\|_E \, \|\Pi^0_p v_h\|_E \,\right] \\
&\leq C \left(1 + |w_h|_{1,\Omega} \right) |v_h|_{1,\Omega} \qquad \text{(using Poincar}\acute{e}\text{ inequality).} \qquad (4.2.15)
\end{aligned}$$

Let $K_3 := |\, d_h(\phi_h; w_h, v_h)\,|$. Proceeding similar to the above derivation, we get,

$$\begin{aligned}
K_3 &\leq \sum_{E\in\mathcal{T}_h} \mathcal{Q}(N)\, \delta_E \left[\, \|\nabla\cdot \mathbf{\Pi}^0_{p-1}\epsilon\nabla w_h\|_E \, \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_E + \mathcal{Q}(\lambda)\|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_E \right] \\
&\leq C \sum_{E\in\mathcal{T}_h} \left[\, h_E^{-1}\, c^*_{inv}\, \delta_E\, \epsilon\, \|\mathbf{\Pi}^0_{p-1}\nabla w_h\|_E + \delta_E \,\right] \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_E \\
&\leq C \sum_{E\in\mathcal{T}_h} h_E \left(\|\nabla w_h\|_E + 1 \right) \|\nabla v_h\|_E \qquad \text{(using (4.2.12))} \\
&\leq C\, h \left(|w_h|_{1,\Omega} + 1 \right) |v_h|_{1,\Omega}. \qquad (4.2.16)
\end{aligned}$$

From the estimates (4.2.14)-(4.2.16), we infer that $\sigma_h(\phi_h; w_h, \cdot)$ is a bounded linear functional on V_h^p. □

Using $\sigma_h(\cdot;\cdot,\cdot)$ in Lemma 4.1, we define a continuous operator $M_h : I \times (H^1_0(\Omega) \cap L^\infty(\Omega)) \to H^{-1}(\Omega)$ such that

$$\langle\, M_h(\mu, w_h), v_h \,\rangle := \sigma_h(w_h; w_h, v_h) \quad \forall v_h \in V_h^p. \qquad (4.2.17)$$

Next, we consider a bounded linear discrete inverse Laplace operator $\mathrm{T}_h : H^{-1}(\Omega) \to V_h^p$ solving

$$a_h(\mathrm{T}_h g, v_h) = \epsilon \,\langle\, g, v_h \,\rangle \quad \forall v_h \in V_h^p. \qquad (4.2.18)$$

Using the operators in (4.2.17) and (4.2.18), we reformulate (4.2.13) equivalently as,

$$\begin{cases} \text{find } u_h \in V_h^p \text{ such that} \\ F_h(\mu, u_h) := u_h + \mathrm{T}_h M_h(\mu, u_h) = 0. \end{cases} \qquad (4.2.19)$$

Remark 4.2. Let $Y_h = Y$, where Y is as in remark 4.1. We know $H^{-1}(\Omega) \subset Y$. It holds $\mathrm{T}_h \in \mathrm{L}(Y_h, V_h^p)$. Let $\|\cdot\|_*$ be the norm on Y_h and is defined as

$$\|\cdot\|_* := \sup_{0\neq z_h \in V_h^k} \frac{\langle \cdot, z_h \rangle}{|z_h|_1}. \qquad (4.2.20)$$

Remark 4.3. Using (4.1.2), assumptions 4.2-4.4 and (4.2.12), we prove the existence and uniqueness of a branch of discrete solutions to (4.2.19) in a neighbourhood of $\mathcal{B}$ by utilizing *Theorem 3.8* (IV, sec.3.4) in [80].

4.3 A Priori estimates

In this section, we state the inverse inequality and interpolation estimates that are useful for our estimation. Then, we prove the auxiliary results that will be used in our error analysis. Hereafter, we assume E is convex $\forall E \in \mathcal{T}_h$ and $\forall \mu \in I,\, u_\mu \in H^{s+1}(\Omega),\ s \in \mathbb{N}$. For simplicity we denote $b_h(\{\phi_h, \phi_h\}\,; w_h, v_h)$ by the notation $b_h(\phi_h; w_h, v_h)$. For convenience, we shall denote $u := u_\mu$ and $u_I := u_{I,\mu}$.

We recall the inverse inequality in [31] i.e., For any $w_h \in V_h^p$ and $\forall E \in \mathcal{T}_h, \quad \exists$ a constant $c_{inv} > 0$ (independent of $h_E,\ E,\ w_h$) such that

$$\|\nabla \cdot \epsilon \nabla w_h\|_E \leq\, c_{inv}\, h_E^{-1}\, \|\epsilon \nabla w_h\|_E. \tag{4.3.1}$$

The following local polynomial interpolation estimates (see Lemma 5.1 in [12]) are considered i.e., for all $E \in \mathcal{T}_h$ and any $\psi \in H^s(E)$,

$$\|\psi - \Pi_p^0 \psi\|_{m,E} \leq C\, h_E^{s-m}\, |\psi|_{s,E} \quad m, s \in \mathbb{N} \cup \{0\},\ m \leq s \leq k+1. \tag{4.3.2}$$

$$\|\psi - \Pi_p^\nabla \psi\|_{m,E} \leq C\, h_E^{s-m}\, |\psi|_{s,E} \quad m, s \in \mathbb{N},\ m \leq s \leq k+1,\ s \geq 1. \tag{4.3.3}$$

The virtual interpolation estimate below is found in [13]. For $0 \leq s \leq k$ and for every $\psi \in H^{1+s}(\Omega)$, there exists $\psi_I \in V_h^p$ satisfying

$$\|\psi - \psi_I\|_\Omega + h\, |\psi - \psi_I|_{1,\Omega} \leq C\, h^{1+s}\, |\psi|_{1+s,\Omega}. \tag{4.3.4}$$

The results that appear in subsequent remarks will be used throughout the analysis.

Remark 4.4. On a bounded Lipschitz domain $\mathcal{D} \in \mathbb{R}^2$ we have the compact Sobolev embedding $H^1(\mathcal{D}) \hookrightarrow L^p(\mathcal{D}),\ 2 \leq p < \infty$. That is, for any $w \in H^1(\mathcal{D})$ we have

$$\|w\|_{L^p(\mathcal{D})} \quad \leq \quad C\, \|w\|_{1,\mathcal{D}}. \tag{4.3.5}$$

Then for $\Pi_p^0 w \in \mathbb{P}_k(E) \subset H^1(E)$ and using the estimates (4.3.5), ((2.44) in [82]) we have

$$\|\Pi_p^0 w\|_{L^p(E)} \quad \leq \quad C\, \|\Pi_p^0 w\|_{1,E} \leq C\, \|w\|_{1,E}. \tag{4.3.6}$$

Remark 4.5. Let $(\cdot, u) \in \mathcal{B}$ and u_I be the virtual interpolant of u as in (4.3.4). The following estimate holds for each $E \in \mathcal{T}_h$:

$$\begin{aligned} \|\Pi_p^0 u\|_{L^\infty(E)} &\leq C\left(h_E^{-1} \|\Pi_p^0 u\|_E + |\Pi_p^0 u|_{1,E} + h_E |\Pi_p^0 u|_{2,E} \right) \quad (\text{ use } (2.8,[82]) \text{ }) \\ &\leq C\left(h_E^{-1} \|u\|_E + |u|_{1,E} + h_E |\Pi_p^0 u|_{2,E} \right) \quad (\text{ use } (2.44,[82]) \text{ }) \\ &\leq C\left(2\,|u|_{1,E} + c_{inv}\,|u|_{1,E} \right) \quad (\text{ use Poincaré, (lemma 10,[49]), (2.44,[82]) }) \\ &\leq C\,|u|_{1,E} \leq C\,\lambda \qquad (\text{ use assumption 3 }) \end{aligned} \tag{4.3.7}$$

Similarly,

$$\begin{aligned} \|\Pi_p^0 u_I\|_{L^\infty(E)} &\leq C\left(h_E^{-1} \|\Pi_p^0 u_I\|_E + |\Pi_p^0 u_I|_{1,E} + h_E |\Pi_p^0 u_I|_{2,E} \right) \\ &\leq C\left(h_E^{-1} (\|u_I - u\|_E + \|u\|_E) + (1 + c_{inv})\,(|u_I - u|_{1,E} + |u|_{1,E}) \right) \\ &\leq C\,(3 + c_{inv})\,|u|_{1,E} \quad (\text{ use Poincaré, (lemma 10,[49]), (2.44,[82]) , (4.3.4) }) \\ &\leq C\,|u|_{1,E} \leq C\,\lambda \qquad (\text{ use assumption 3 }) \end{aligned} \tag{4.3.8}$$

Lemma 4.2. *For any $(\cdot, u) \in \mathcal{B}$ and its virtual interpolant $u_I \in V_h^p$, using* (4.1.2)*, assumptions 4.2-4.4 and* (4.2.12)*, we obtain,*

$$\mu\, b_h(u_I; u_I, v_h) - \mu\, b(u; u, v_h) \leq \mathcal{C}_1\, h^s\, |v_h|_{1,\Omega} \quad \forall\, v_h \in V_h^k, \tag{4.3.9}$$

where $\mathcal{C}_1 := \mu_2\, [\, \mathcal{Q}(C\lambda)\,]^2\, C \left[\, |u|_{1,\Omega} + 3\,\delta + \delta\, |u|_{2,\Omega} \right] |u|_{1+s,\Omega}$.

Proof. Let $K_1 := \mu\, b_h(u_I; u_I, v_h) - \mu\, b(u; u, v_h)$. Then,

$$\begin{aligned} K_1 = \mu \sum_{E \in \mathcal{T}_h} \Big\{ &\left(\boldsymbol{\beta}(\Pi_p^0 u_I) \cdot \boldsymbol{\Pi}_{p-1}^0 \nabla u_I,\ \delta_E\, \boldsymbol{\beta}(\Pi_p^0 u_I) \cdot \boldsymbol{\Pi}_{p-1}^0 \nabla v_h \right)_E \\ &- \left(\boldsymbol{\beta}(u) \cdot \nabla u,\ \delta_E\, \boldsymbol{\beta}(u) \cdot \nabla v_h \right)_E \Big\} \\ &+ \mu \sum_{E \in \mathcal{T}_h} \delta_E\, \tilde{\mathbb{S}}^2\, S_E((I - \Pi_k^\nabla) u_I, (I - \Pi_k^\nabla) v_h) \;=\; K_{11} \;+\; K_{12}. \end{aligned} \tag{4.3.10}$$

Let us define,

$$\begin{aligned} \bar{K} \;=\; \mu \sum_{E \in \mathcal{T}_h} \Big\{ &\left([\, \boldsymbol{\beta}(\Pi_p^0 u) + \boldsymbol{\beta}(u)\,] \cdot \boldsymbol{\Pi}_{p-1}^0 \nabla u_I,\ \delta_E\, \boldsymbol{\beta}(\Pi_p^0 u_I) \cdot \boldsymbol{\Pi}_{p-1}^0 \nabla v_h \right)_E \\ &+ \left(\boldsymbol{\beta}(\Pi_p^0 u) \cdot [\, \boldsymbol{\Pi}_{p-1}^0 \nabla u + \nabla u\,],\ \delta_E\, \boldsymbol{\beta}(\Pi_p^0 u_I) \cdot \boldsymbol{\Pi}_{p-1}^0 \nabla v_h \right)_E \\ &+ \left(\boldsymbol{\beta}(u) \cdot \nabla u,\ \delta_E\, [\, \boldsymbol{\beta}(\Pi_p^0 u) + \boldsymbol{\beta}(u)\,] \cdot \boldsymbol{\Pi}_{p-1}^0 \nabla v_h \right)_E \Big\}. \end{aligned}$$

Adding and subtracting $\bar{K}$ to K_{11} in (4.3.10) we get,

$$
\begin{aligned}
K_{11} &:= \mu \sum_{E\in\mathcal{T}_h} \Big\{ \Big(\big(\boldsymbol{\beta}(\Pi^0_p u_I) - \boldsymbol{\beta}(\Pi^0_p u) \big) \cdot \mathbf{\Pi}^0_{p-1}\nabla u_I,\ \delta_E\, \boldsymbol{\beta}(\Pi^0_p u_I)\cdot \mathbf{\Pi}^0_{p-1}\nabla v_h \Big)_E \\
&\quad + \Big(\big(\boldsymbol{\beta}(\Pi^0_p u) - \boldsymbol{\beta}(u) \big) \cdot \mathbf{\Pi}^0_{p-1}\nabla u_I,\ \delta_E\, \boldsymbol{\beta}(\Pi^0_p u_I)\cdot \mathbf{\Pi}^0_{p-1}\nabla v_h \Big)_E \\
&\quad + \Big(\boldsymbol{\beta}(\Pi^0_p u) \cdot \big[\mathbf{\Pi}^0_{p-1}\nabla (u_I - u) + (\mathbf{\Pi}^0_{p-1}\nabla u - \nabla u) \big],\ \delta_E\, \boldsymbol{\beta}(\Pi^0_p u_I)\cdot \mathbf{\Pi}^0_{p-1}\nabla v_h \Big)_E \\
&\quad + \Big(\boldsymbol{\beta}(u)\cdot\nabla u,\ \delta_E \big[\big(\boldsymbol{\beta}(\Pi^0_p u_I) - \boldsymbol{\beta}(\Pi^0_p u) \big) + \big(\boldsymbol{\beta}(\Pi^0_p u) - \boldsymbol{\beta}(u) \big) \big] \cdot \mathbf{\Pi}^0_{p-1}\nabla v_h \Big)_E \\
&\quad + \Big(\boldsymbol{\beta}(u)\cdot\nabla u,\ \delta_E\, \boldsymbol{\beta}(u) \cdot \big[\mathbf{\Pi}^0_{p-1}\nabla v_h - \nabla v_h \big] \Big)_E \Big\} = \iota_1 + \iota_2 + \iota_3 + \iota_4. \quad (4.3.11)
\end{aligned}
$$

Using mean value theorem (MVT), remark 4.5, (4.1.2), Cauchy-Schwarz inequality, generalised Hölder's inequality, (4.3.6) and (4.3.5) we get

$$
\begin{aligned}
\iota_1 &\le \mu \sum_{E\in\mathcal{T}_h} [\mathcal{Q}(C\lambda)]^2\, \delta_E \big[\|\Pi^0_p(u_I - u)\|_{L^3(E)} + \|\Pi^0_p u - u\|_{L^3(E)} \big]\, |u_I|_{1,E}\, \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{L^6(E)} \\
&\le \mu \sum_{E\in\mathcal{T}_h} [\mathcal{Q}(C\lambda)]^2\, \delta_E \big[\|u_I - u\|_{1,E} + \|\Pi^0_p u - u\|_{1,E} \big]\, |u_I|_{1,E}\, \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{1,E} \\
&\le \mu \sum_{E\in\mathcal{T}_h} [\mathcal{Q}(C\lambda)]^2\, \delta_E \big[\|u_I - u\|_{1,E} + C\, h^s_E\, |u|_{1+s,E} \big]\, |u_I|_{1,E}\, h_E^{-1}\, |v_h|_{1,E} \ \text{(use (4.3.2), (2.6,[82]))} \\
&\le \mu\, [\mathcal{Q}(C\lambda)]^2 \big[\|u_I - u\|_{1,\Omega} + C\, h^s\, |u|_{1+s,\Omega} \big]\, |u_I|_{1,\Omega}\, |v_h|_{1,\Omega} \ \text{(use (4.2.12) \& Hölder's ineq.)} \\
&\le \mu\, C\, [\mathcal{Q}(C\lambda)]^2\, h^s\, |u|_{1+s,\Omega}\, |u|_{1,\Omega}\, |v_h|_{1,\Omega}. \quad \text{(use (4.3.4))} \qquad (4.3.12)
\end{aligned}
$$

Using remark 4.5, (4.1.2), (4.3.2), Hölder's inequality and (4.3.4), we get,

$$
\begin{aligned}
\iota_2 &\le \mu \sum_{E\in\mathcal{T}_h} [\mathcal{Q}(C\lambda)]^2\, \delta_E \big[|u_I - u|_{1,E} + \|\mathbf{\Pi}^0_{p-1}\nabla u - \nabla u\|_E \big]\, |v_h|_{1,E} \\
&\le \mu \sum_{E\in\mathcal{T}_h} [\mathcal{Q}(C\lambda)]^2\, \delta_E \big[|u_I - u|_{1,E} + |\Pi^\nabla_p u - u|_{1,E} \big]\, |v_h|_{1,E} \quad \text{(use (5.22,[12]))} \\
&\le \mu \sum_{E\in\mathcal{T}_h} [\mathcal{Q}(C\lambda)]^2\, \delta_E \big[|u_I - u|_{1,E} + C\, h^s_E |u|_{1+s,E} \big]\, |v_h|_{1,E} \\
&\le \mu\, C\, [\mathcal{Q}(C\lambda)]^2\, \delta\, h^s\, |u|_{1+s,\Omega}\, |v_h|_{1,\Omega}. \qquad (4.3.13)
\end{aligned}
$$

Using MVT, remark 4.5, (4.1.2), Cauchy-Schwarz inequality, generalised Hölder's inequality and (4.3.6), we get,

$$
\begin{aligned}
\iota_3 &\le \mu \sum_{E\in\mathcal{T}_h} [\mathcal{Q}(C\lambda)]^2\, \delta_E\, \|\nabla u\|_{L^6(E)} \big[\|\Pi^0_p(u_I - u)\|_{L^3(E)} + \|\Pi^0_p u - u\|_{L^3(E)} \big]\, \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_E \\
&\le \mu \sum_{E\in\mathcal{T}_h} [\mathcal{Q}(C\lambda)]^2\, \delta_E\, \|\nabla u\|_{1,E} \big[\|u_I - u\|_{1,E} + \|\Pi^0_p u - u\|_{1,E} \big]\, |v_h|_{1,E} \\
&\le \mu \sum_{E\in\mathcal{T}_h} [\mathcal{Q}(C\lambda)]^2\, \delta_E\, |u|_{2,E} \big[\|u_I - u\|_{1,E} + C\, h^s_E\, |u|_{1+s,E} \big]\, |v_h|_{1,E}
\end{aligned}
$$

Applying Hölder's inequality (4.3.4), we get

$$\iota_3 \;\leq\; \mu\, C\, [\mathcal{Q}(C\lambda)]^2\, \delta\, |u|_{2,\Omega} h^s\, |u|_{1+s,\Omega}\, |v_h|_{1,\Omega}. \quad (\text{ use }) \tag{4.3.14}$$

Next we estimate,

$$\begin{aligned}
\iota_4 \;&=\; \mu \sum_{E\in\mathcal{T}_h} \big(\, \boldsymbol{\beta}(u)\cdot\nabla u,\; \delta_E\, \boldsymbol{\beta}(u)\cdot \big[\, \boldsymbol{\Pi}^0_{p-1}\nabla v_h - \nabla v_h \,\big]\,\big)_E \\
&\leq\; \mu\, [\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h} \big(\, \nabla u,\; \delta_E \big[\, \boldsymbol{\Pi}^0_{p-1}\nabla v_h - \nabla v_h \,\big]\,\big)_E \\
&=\; \mu\, [\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h} \big(\, \nabla u - \boldsymbol{\Pi}^0_{p-1}\nabla u,\; \delta_E\, \nabla v_h \,\big)_E \\
&\leq\; \mu\, [\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h} |u - \Pi^\nabla_p u|_{1,E}\, \delta_E\, |v_h|_{1,E} \quad (\text{ use Cauchy-Schwarz ineq., (5.22, [12]) }) \\
&\leq\; \mu\, [\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h} C\, h_E^s\, |u|_{1+s,E}\, \delta_E\, |v_h|_{1,E} \quad (\text{ use (4.3.3) }) \\
&\leq\; \mu\, [\mathcal{Q}(\lambda)]^2\, \delta\, C\, h^s\, |u|_{1+s,\Omega}\, |v_h|_{1,\Omega}. \qquad (\text{ use Hölder's inequality })
\end{aligned} \tag{4.3.15}$$

Using (4.2.11) and (4.2.5) we obtain

$$K_{12} \;\leq\; \mu \sum_{E\in\mathcal{T}_h} \delta_E\, [\mathcal{Q}(\lambda)]^2\, |u_I - \Pi^\nabla_p u_I|_{1,E}\; |v_h - \Pi^\nabla_p v_h|_{1,E}.$$

Note the inequality,

$$\begin{aligned}
|u_I - \Pi^\nabla_p u_I|_{1,E} \;&\leq\; |u_I - u|_{1,E} + |u - \Pi^\nabla_p u|_{1,E} + |\Pi^\nabla_p (u - u_I)|_{1,E} \\
&\leq\; 2\, |u_I - u|_{1,E} + |u - \Pi^\nabla_p u|_{1,E}.
\end{aligned} \tag{4.3.16}$$

Thus

$$\begin{aligned}
K_{12} \;&\leq\; \mu \sum_{E\in\mathcal{T}_h} \delta_E\, [\mathcal{Q}(\lambda)]^2\, [\, 2\, |u_I - u|_{1,E} + |u - \Pi^\nabla_p u|_{1,E}\,]\, |v_h - \Pi^\nabla_p v_h|_{1,E} \\
&\leq\; \mu \sum_{E\in\mathcal{T}_h} \delta_E\, [\mathcal{Q}(\lambda)]^2\, [\, 2\, |u_I - u|_{1,E} + C\, h_E^s\, |u|_{1+s,E}\,]\, |v_h|_{1,E} \quad (\text{ use (4.3.3) }) \\
&\leq\; \mu\, \delta\, [\mathcal{Q}(\lambda)]^2\, [\, 2\, |u_I - u|_{1,\Omega} + C\, h^s\, |u|_{1+s,\Omega}\,]\, |v_h|_{1,\Omega} \quad (\text{ use Hölder's ineq. }) \\
&\leq\; \mu\, \delta\, [\mathcal{Q}(\lambda)]^2\, C\, h^s\, |u|_{1+s,\Omega}\, |v_h|_{1,\Omega}. \quad (\text{ use (4.3.4) })
\end{aligned} \tag{4.3.17}$$

Combining the results (4.3.12), (4.3.13), (4.3.14), (4.3.15), and (4.3.17) we get the required estimate (4.3.9). □

Lemma 4.3. *For any* $(\cdot, u) \in \mathcal{B}$ *and its virtual interpolant* $u_I \in V_h^p$, *using* (4.1.2), *assumptions 4.2-4.4 and* (4.2.12), *we obtain,*

$$\mu\, c_h(u_I; u_I, v_h) - \mu\, c(u; u, v_h) \leq \mathcal{C}_2\, h^s\, |v_h|_{1,\Omega} \qquad \forall\, v_h \in V_h^k, \tag{4.3.18}$$

where $\mathcal{C}_2 := \mu_2\, \mathcal{Q}(C\lambda)\, C\, [\, |u|_{1,\Omega} + h + 1\,]\, |u|_{1+s,\Omega}$.

Proof. Let us denote $K_2 := \mu \left[\, c_h(u_I; u_I, v_h) - c(u; u, v_h)\,\right]$. We have,

$$\begin{aligned} K_2 &= \mu \sum_{E\in\mathcal{T}_h} \left[\, (\, \boldsymbol{\beta}(\Pi_p^0 u_I) \cdot \boldsymbol{\Pi}_{p-1}^0 \nabla u_I,\, \Pi_p^0 v_h\,)_E - (\, \beta(u)\cdot\nabla u,\, v_h\,)_E\,\right] \\ &\quad + \mu \sum_{E\in\mathcal{T}_h} \left[\, (\, r(\Pi_p^0 u_I), \Pi_p^0 v_h\,)_E - (\, r(u), v_h\,)_E\,\right] \;=\; K_{21} \,+\, K_{22}. \end{aligned}$$

Adding and subtracting the following terms to K_{21}
$\mu \sum_{E\in\mathcal{T}_h} \big(\, [\, \boldsymbol{\beta}(\Pi_p^0 u) + \boldsymbol{\beta}(u)\,] \cdot \boldsymbol{\Pi}_{p-1}^0 \nabla u \,+\, \boldsymbol{\beta}(u) \cdot [\, \boldsymbol{\Pi}_{p-1}^0 \nabla u - \nabla u\,],\ \Pi_p^0 v_h\,\big)_E$ we get

$$\begin{aligned} K_{21} &= \mu \sum_{E\in\mathcal{T}_h} \Big\{ \Big(\big[\, (\, \boldsymbol{\beta}(\Pi_p^0 u_I) - \boldsymbol{\beta}(\Pi_p^0 u)\,) + (\, \boldsymbol{\beta}(\Pi_p^0 u) - \boldsymbol{\beta}(u)\,)\,\big] \cdot \boldsymbol{\Pi}_{p-1}^0 \nabla u_I,\ \Pi_p^0 v_h\,\Big)_E \\ &\quad + \Big(\boldsymbol{\beta}(u) \cdot \big[\, \boldsymbol{\Pi}_{p-1}^0 \nabla (u_I - u) \,+\, \boldsymbol{\Pi}_{p-1}^0 \nabla u - \nabla u)\big],\ \Pi_p^0 v_h\Big)_E \\ &\quad + \Big(\boldsymbol{\beta}(u)\cdot\nabla u,\ \Pi_p^0 v_h - v_h\,\Big)_E \Big\} = l_1 + l_2 + l_3. \end{aligned} \tag{4.3.19}$$

Using MVT, remark 4.5, (4.1.2), Cauchy-Schwarz inequality, generalised Hölder's inequality, (4.3.6) and (4.3.5), we get,

$$\begin{aligned} l_1 &\leq \mu \sum_{E\in\mathcal{T}_h} \mathcal{Q}(C\lambda)\, [\, \|\Pi_p^0(u_I - u)\|_{L^6(E)} + \|\Pi_p^0 u - u\|_{L^6(E)}\,]\, \|\boldsymbol{\Pi}_{p-1}^0 \nabla u_I\|_E\, \|\Pi_p^0 v_h\|_{L^3(E)} \\ &\leq \mu \sum_{E\in\mathcal{T}_h} \mathcal{Q}(C\lambda)\, [\, \|u_I - u\|_{1,E} + \|\Pi_p^0 u - u\|_{1,E}\,]\, |u_I|_{1,E}\, \|v_h\|_{1,E} \\ &\leq \mu\, \mathcal{Q}(C\lambda)\, [\, \|u_I - u\|_{1,\Omega} + C\, h^s\, |u|_{1+s,\Omega}\,]\, |u_I|_{1,\Omega}\, |v_h|_{1,\Omega} \qquad \text{(use (4.3.2), Hölder's ineq.)} \\ &\leq \mu\, C\, \mathcal{Q}(C\lambda)\, h^s\, |u|_{1+s,\Omega}\, |u|_{1,\Omega}\, |v_h|_{1,\Omega}. \qquad \text{(use (4.3.4))} \end{aligned} \tag{4.3.20}$$

Using assumption 4.3, (4.1.2) and Cauchy-Schwarz inequality, we get,

$$\begin{aligned} l_2 &\leq \mu \sum_{E\in\mathcal{T}_h} \mathcal{Q}(C\lambda)\, [\, |u_I - u|_{1,E} + \|\boldsymbol{\Pi}_{p-1}^0 \nabla u - \nabla u\|_E\,]\, \|v_h\|_E \\ &\leq \mu \sum_{E\in\mathcal{T}_h} \mathcal{Q}(C\lambda)\, [\, |u_I - u|_{1,E} + |\Pi_p^\nabla u - u|_{1,E}\,]\, \|v_h\|_E \qquad \text{(use (5.22,[12]))} \\ &\leq \mu \sum_{E\in\mathcal{T}_h} \mathcal{Q}(C\lambda)\, [\, |u_I - u|_{1,E} + C\, h_E^s\, |u|_{1+s,E}\,]\, \|v_h\|_E \qquad \text{(use (4.3.3))} \\ &\leq \mu\, \mathcal{Q}(C\lambda)\, C\, h^s\, |u|_{1+s,\Omega}\, |v_h|_{1,\Omega}. \qquad \text{(use (4.3.4), Poincaré inequality)} \end{aligned} \tag{4.3.21}$$

Now we have $l_3 := \mu \sum_{E\in\mathcal{T}_h} \sum_{i=1,2} \big(\beta_i(u)\partial_{x_i} u,\ \Pi_p^0 v_h - v_h \big)_E$, where $\partial_{x_i} u := \partial u/\partial x_i$.

$$
\begin{aligned}
t_1 := \big(\beta_i(u)\, \partial_{x_i} u,\ \Pi_p^0 v_h - v_h \big)_E \ &\leq\ \mathcal{Q}(C\lambda) \big(\partial_{x_i} u - \Pi_{k-1}^0 \partial_{x_i} u,\ \Pi_p^0 v_h - v_h \big)_E \\
&\leq\ \mathcal{Q}(C\lambda)\, \|\partial_{x_i} u - \Pi_{k-1}^0 \partial_{x_i} u\|_E\, \|\Pi_p^0 v_h - v_h\,\|_E
\end{aligned}
$$

Using (4.3.2), we get,

$$
t_1 \ \leq\ C\, \mathcal{Q}(C\lambda)\, h_E^s\, |\partial_{x_i} u|_{s,E}\, h_E\, |v_h|_1 \ \leq\ C\, \mathcal{Q}(C\lambda)\, h_E^{1+s}\, |u|_{1+s,E}\, |v_h|_1.
$$

Then, substituting t_1 in l_3 and applying H$\ddot{o}$lder's inequality, we obtain,

$$
l_3 \ \leq\ \mu\, C\, \mathcal{Q}(C\lambda)\, h^{1+s}\, |u|_{1+s,\Omega}\, |v_h|_{1,\Omega}. \tag{4.3.22}
$$

Adding and subtracting the terms $\mu \sum_{E\in\mathcal{T}_h} (\, r(\Pi_p^0 u) + r(u), \Pi_p^0 v_h \,)_E$ to K_{22}, we get,

$$
K_{22} = \mu \sum_{E\in\mathcal{T}_h} \big[\, (\, r(\Pi_p^0 u_I) - r(\Pi_p^0 u) + r(\Pi_p^0 u) - r(u),\ \Pi_p^0 v_h \,)_E +\ (r(u) - r(0),\ (\Pi_p^0 - I) v_h)_E \,\big].
$$

Using MVT, Cauchy-Schwarz inequality, Remark 4.5, (4.1.2) and (4.3.2), we get,

$$
\begin{aligned}
K_{22} \ &\leq\ \mu\, \mathcal{Q}(C\lambda) \sum_{E\in\mathcal{T}_h} \big[\, (\, \Pi_p^0(u_I - u) + (\Pi_p^0 u - u), \Pi_p^0 v_h \,)_E + (u,\ \Pi_p^0 v_h - v_h)_E \,\big] \\
&\leq\ \mu\, \mathcal{Q}(C\lambda) \sum_{E\in\mathcal{T}_h} \big[\, (\, \Pi_p^0(u_I - u) + (\Pi_p^0 u - u), \Pi_p^0 v_h \,)_E + (u - \Pi_p^0 u,\ v_h)_E \,\big] \\
&\leq\ \mu\, \mathcal{Q}(C\lambda) \sum_{E\in\mathcal{T}_h} \big(\|u_I - u\|_E + 2\, \|\Pi_p^0 u - u\|_E \big)\, \|v_h\|_E \\
&\leq\ \mu\, \mathcal{Q}(C\lambda) \sum_{E\in\mathcal{T}_h} \big(\|u_I - u\|_E + C\, h_E^{1+s} |u|_{1+s,E} \big)\, \|v_h\|_E.
\end{aligned}
$$

Using H$\ddot{o}$lder's inequality, (4.3.4) and Poincar$\acute{e}$ inequality, we get,

$$
K_{22} \ \leq\ \mu\, C\, \mathcal{Q}(C\lambda)\, h^{1+s}\, |u|_{1+s,\Omega}\, |v_h|_{1,\Omega}. \tag{4.3.23}
$$

Adding (4.3.20), (4.3.21), (4.3.22) and (4.3.23), we obtain the assertion (4.3.18). □

Lemma 4.4. *For any* $(\cdot, u) \in \mathcal{B}$ *and its virtual interpolant* $u_I \in V_h^p$*, using* (4.1.2)*, assumptions 4.2-4.4 and* (4.2.12)*, we obtain,*

$$
\mu\, d_h(u_I; u_I, v_h) - \mu\, d(u; u, v_h) \ \leq \mathcal{C}_3\, h^s\, |v_h|_{1,\Omega} \qquad \forall\, v_h \in V_h^k, \tag{4.3.24}
$$

where $\mathcal{C}_3 := \mu_2\,\mathcal{Q}(C\lambda)\,C\,\big[\,2h + |u|_{1,\Omega} + 2\,\mathcal{Q}(C\lambda)\,\delta\,h\,\big]\,|u|_{1+s,\Omega}$.

Proof. Let us denote $K_3 := \mu\,\big[\,d_h(u_I;u_I,v_h) - d(u;u,v_h)\,\big]$. We have

$$\begin{aligned} K_3 &= \mu\sum_{E\in\mathcal{T}_h}\{\,\big(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla u_I,\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E - (\,-\epsilon\,\Delta u,\ \delta_E\,\boldsymbol{\beta}(u)\cdot\nabla v_h\,)_E\,\} \\ &\quad +\mu\sum_{E\in\mathcal{T}_h}\{\,\big(\,r(\Pi^0_p u_I),\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E - (\,r(u),\ \delta_E\,\boldsymbol{\beta}(u)\cdot\nabla v_h\,)_E\,\} = K_{31} + K_{32}. \end{aligned}$$

Let us define, $\hat{K} = \mu\sum_{E\in\mathcal{T}_h}\big\{\,\big(-\nabla\cdot\epsilon\,(\,\nabla u_I + \nabla u\,),\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E$

$$+\ \big(-\nabla\cdot\epsilon\,\nabla u,\ \delta_E\,[\,\boldsymbol{\beta}(\Pi^0_p u) + \boldsymbol{\beta}(u)\,]\cdot\nabla v_h\,\big)_E\,\big\}.$$

Adding and subtracting $\hat{K}$ to K_{31} , we get,

$$\begin{aligned} K_{31} &:= \mu\sum_{E\in\mathcal{T}_h}\big(-\nabla\cdot\epsilon\,(\,\mathbf{\Pi}^0_{p-1}\nabla u_I - \nabla u_I\,) + \nabla\cdot\epsilon\,\nabla(u-u_I),\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E \\ &\quad +\mu\sum_{E\in\mathcal{T}_h}\big(-\nabla\cdot\epsilon\,\nabla u,\ \delta_E\,\big[\,(\,\boldsymbol{\beta}(\Pi^0_p u_I) - \boldsymbol{\beta}(\Pi^0_p u)\,) + (\,\boldsymbol{\beta}(\Pi^0_p u) - \boldsymbol{\beta}(u)\,)\,\big]\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E \\ &\quad +\mu\sum_{E\in\mathcal{T}_h}\big(-\nabla\cdot\epsilon\,\nabla u,\ \delta_E\,\boldsymbol{\beta}(u)\cdot[\,\mathbf{\Pi}^0_{p-1}\nabla v_h - \nabla v_h\,]\,\big)_E = j_1 + j_2 + j_3. \end{aligned} \tag{4.3.25}$$

Using Remark 4.5, (4.1.2), Cauchy-Schwarz inequality, (4.3.1), we get,

$$\begin{aligned} j_1 &\le \mu\,\mathcal{Q}(C\lambda)\sum_{E\in\mathcal{T}_h}\big[\,\|\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla u_I - \nabla u_I\,)\|_E + \|\nabla\cdot\epsilon\,\nabla(u-u_I)\|_E\,\big]\ \delta_E\,\|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_E \\ &\le \mu\,\mathcal{Q}(C\lambda)\sum_{E\in\mathcal{T}_h}\delta_E\,\epsilon\,c_{inv}\,h_E^{-1}\,\big[\,\|\mathbf{\Pi}^0_{p-1}\nabla u_I - \nabla u_I\,)\|_E + \|\nabla(u-u_I)\|_E\,\big]\,|v_h|_{1,E} \\ &\le \mu\,\mathcal{Q}(C\lambda)\sum_{E\in\mathcal{T}_h}C\,h_E\,\big[\,|\Pi^\nabla_k u_I - u_I|_{1,E} + |u-u_I|_{1,E}\,\big]\,|v_h|_{1,E} \quad \text{(use (4.2.12), (5.22, [12]))} \\ &\le \mu\,\mathcal{Q}(C\lambda)\sum_{E\in\mathcal{T}_h}C\,h_E\,\big[\,|\Pi^\nabla_k u - u|_{1,E} + 3|u-u_I|_{1,E}\,\big]\,|v_h|_{1,E} \quad \text{(use (4.3.16))} \\ &\le \mu\,\mathcal{Q}(C\lambda)\,C\,h^{1+s}\,|u|_{1+s,\Omega}\,|v_h|_{1,\Omega}. \quad \text{(use Hölder's inequality, (4.3.2), (4.3.3))} \end{aligned} \tag{4.3.26}$$

Using MVT, Remark 4.5, (4.1.2), generalised Hölder's inequality, (4.3.5), (4.3.6) we get,

$$\begin{aligned} j_2 &\le \mu\,\mathcal{Q}(C\lambda)\sum_{E\in\mathcal{T}_h}\delta_E\,\|\nabla\cdot\epsilon\,\nabla u\|_E\,\big[\,\|\Pi^0_p(u_I-u)\|_{L^3(E)} + \|\Pi^0_p u - u\|_{L^3(E)}\,\big]\,\|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{L^6(E)} \\ &\le \mu\,\mathcal{Q}(C\lambda)\sum_{E\in\mathcal{T}_h}\delta_E\,\|\nabla\cdot\epsilon\,\nabla u\|_E\,\big[\,\|u_I - u\|_{1,E} + \|\Pi^0_p u - u\|_{1,E}\,\big]\,\|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{1,E} \\ &\le \mu\,C\,\mathcal{Q}(C\lambda)\sum_{E\in\mathcal{T}_h}\delta_E\,\epsilon\,h_E^{-2}|u|_{1,E}\,\big[\,\|u_I - u\|_{1,E} + \|\Pi^0_p u - u\|_{1,E}\,\big]\,|v_h|_{1,E} \text{(use (4.3.1), (2.6, [82]))} \\ &\le \mu\,C\,\mathcal{Q}(C\lambda)\sum_{E\in\mathcal{T}_h}|u|_{1,E}\,\big[\,\|u_I - u\|_{1,E} + C\,h_E^s\,|u|_{1+s,E}\,\big]\,|v_h|_{1,E} \text{(use (4.2.12), (4.3.2))} \\ &\le \mu\,C\,\mathcal{Q}(C\lambda)\,h^s\,|u|_{1,\Omega}\,|u|_{1+s,\Omega}\,|v_h|_{1,\Omega}. \quad \text{(use Hölder's inequality, (4.3.4))} \end{aligned} \tag{4.3.27}$$

Next $j_3 = \mu \sum_{E\in\mathcal{T}_h} \underbrace{\big(-\nabla\cdot\epsilon\,\nabla u,\ \delta_E\,\boldsymbol{\beta}(u)\cdot[\,\boldsymbol{\Pi}^0_{p-1}\nabla v_h - \nabla v_h\,]\big)_E}_{:=t_2}$. We have,

$$
\begin{aligned}
t_2 &= \sum_{i=1,2} \big(\beta_i(u)\,\epsilon\,\Delta u,\ \delta_E\,[\,\partial_{x_i} v_h - \Pi^0_{k-1}\partial_{x_i} v_h\,]\big)_E \\
&\le \mathcal{Q}(C\lambda) \sum_{i=1,2} \epsilon\,\big(\Delta u - \Pi^0_{k-1}\Delta u,\ \delta_E\,\partial_{x_i} v_h\big)_E \\
&\le \mathcal{Q}(C\lambda) \sum_{i=1,2} \delta_E\,\epsilon\,\|\Delta u - \Pi^0_{k-1}\Delta u\|_E\,\|\partial_{x_i} v_h\|_E \quad \text{(use Cauchy-Schwarz inequality)} \\
&\le \mathcal{Q}(C\lambda) \sum_{i=1,2} h_E^2\,C\,h_E^{1-s}\,|\Delta u|_{1-s,E}\,\|\partial_{x_i} v_h\|_E \quad \text{(use (4.2.12), (4.3.2))} \\
&\le C\,\mathcal{Q}(C\lambda)\,h_E^{1+s}\,|u|_{1+s,E}\,|v_h|_{1,E}. \qquad (4.3.28)
\end{aligned}
$$

Substituting (4.3.28) and using Hölder's inequality, we obtain,

$$
j_3 \le \mu\,C\,\mathcal{Q}(C\lambda)\,h^{1+s}\,|u|_{1+s,\Omega}\,|v_h|_{1,\Omega}. \qquad (4.3.29)
$$

We add and subtract the following term to K_{32}

$$
\mu \sum_{E\in\mathcal{T}_h} \big(r(\Pi^0_p u) + r(u),\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h\big)_E + \big(r(u),\ \delta_E\,[\,\boldsymbol{\beta}(\Pi^0_p u) + \boldsymbol{\beta}(u)\,]\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h\big)_E\,.
$$

Then, we obtain,

$$
\begin{aligned}
K_{32} = \mu \sum_{E\in\mathcal{T}_h} \mu \sum_{E\in\mathcal{T}_h} &\big(\,[\,r(\Pi^0_p u_I) - r(\Pi^0_p u)\,] + [\,r(\Pi^0_p u) - r(u)\,],\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h\big)_E \\
+\mu \sum_{E\in\mathcal{T}_h} &\big(r(u),\ \delta_E\,[\,(\,\boldsymbol{\beta}(\Pi^0_p u_I) - \boldsymbol{\beta}(\Pi^0_p u)\,) + (\,\boldsymbol{\beta}(\Pi^0_p u) - \boldsymbol{\beta}(u)\,)\,]\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h\big)_E \\
+\mu \sum_{E\in\mathcal{T}_h} &\big(r(u),\ \delta_E\,\boldsymbol{\beta}(u)\cdot[\,\boldsymbol{\Pi}^0_{p-1}\nabla v_h - \nabla v_h\,]\big)_E = m_1 + m_2 + m_3.
\end{aligned}
$$

Using MVT, Remark 4.5, (4.1.2) and Cauchy-Schwarz inequality, we get,

$$
\begin{aligned}
m_1 + m_2 &\le 2\,\mu\,[\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h} \delta_E\,[\,\|\Pi^0_p(u_I - u)\|_E + \|\Pi^0_p u - u\|_E\,]\,\|\boldsymbol{\Pi}^0_{p-1}\nabla v_h\|_E \\
&\le 2\,\mu\,[\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h} \delta_E\,[\,\|u_I - u\|_E + C\,h_E^{1+s}\,|u|_{1+s,E}\,]\,|v_h|_{1,E} \quad \text{(use (4.3.2))} \\
&\le \mu\,C\,[\mathcal{Q}(\lambda)]^2\,\delta\,h^{1+s}\,|u|_{1+s,\Omega}\,|v_h|_{1,\Omega} \quad \text{(use Hölder's inequality, (4.3.4))} \qquad (4.3.30)
\end{aligned}
$$

Using Assumption 3 (Section 1.3), (4.1.2), $r(0) = 0$, Cauchy-Schwarz inequality, (4.3.2), and Hölder's inequality we obtain,

$$
\begin{aligned}
m_3 &\leq \mu\, \mathcal{Q}(C\,\lambda) \sum_{E\in\mathcal{T}_h} \sum_{i=1,2} \big(r(u) - r(0),\ \delta_E \,[\, \Pi^0_{k-1}\partial_{xi} v_h - \partial_{xi} v_h \,] \big)_E \\
&\leq \mu\, [\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h} \sum_{i=1,2} \big(u - \Pi^0_{k-1} u,\ \delta_E \,[\, \Pi^0_{k-1}\partial_{xi} v_h - \partial_{xi} v_h \,] \big)_E \\
&\leq \mu\, [\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h} \delta_E \sum_{i=1,2} \|u - \Pi^0_{k-1} u\|_E \, \|\Pi^0_{k-1}\partial_{xi} v_h - \partial_{xi} v_h\|_E \\
&\leq \mu\, [\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h} \delta_E \sum_{i=1,2} C\, h_E^s \, |u|_{s,E} \, h_E \, |\partial_{xi} v_h|_{1,E} \\
&\leq \mu\, [\mathcal{Q}(\lambda)]^2 \, C\, \delta\, h^{1+s} \, |u|_{1+s,\Omega} \, |v_h|_{1,\Omega}. \qquad (4.3.31)
\end{aligned}
$$

Adding estimates (4.3.26), (4.3.27), (4.3.29), (4.3.30) and (4.3.31) we obtain the assertion (4.3.24). □

Lemma 4.5. *For any* $(\cdot, u) \in \mathcal{B}$ *and its virtual interpolant* $u_I \in V_h^p$*, using* (4.1.2)*, Assumptions 4.2-4.4 (section 1.3) and* (4.2.12)*, we obtain,*

$$
\|M_h(\mu, u_I(\mu)) - M(\mu, u(\mu))\|_* \leq \mathcal{C}_\star\, h^s, \qquad (4.3.32)
$$

where $\|\cdot\|_*$ *is defined in* (4.2.20) *and* $\mathcal{C}_\star = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ *is a positive constant.*

Proof. For $0 \neq v_h \in V_h$, let $\mathcal{M} := \langle M_h(\mu, u_I(\mu)) - M(\mu, u(\mu)),\, v_h \rangle$. Using the definitions (4.2.17) we have,

$$
\mathcal{M} \;=\; \mu\, \big[\, b_h(u_I; u_I, v_h) - b(u; u, v_h) + c_h(u_I; u_I, v_h) - c(u; u, v_h) + d_h(u_I; u_I, v_h) - d(u; u, v_h) \,\big].
$$

Therefore, Lemma 4.2, Lemma 4.3 and Lemma 4.4 and the definition of $\|\cdot\|_*$ implies the required estimate (4.3.32). □

Now corresponding to M_0 in (4.1.4) we consider an operator $M : I \times (H^1_0(\Omega) \cap L^\infty(\Omega)) \to H^{-1}(\Omega)$ such that

$$
\langle\, M(\mu, w_h), v_h \,\rangle := \mu \big[b(w_h; w_h, v_h) + c(w_h; w_h, v_h) + d(w_h; w_h, v_h) \big] \quad \forall v_h \in V_h^p. \qquad (4.3.33)
$$

Now the Fr*é*chet derivative of the operator M denoted by $DM(\mu, \cdot)$ satisfies, for any $w \in H^1_0(\Omega)$,

$$
\langle DM(\mu, u) w,\ v \rangle := \mu\, \big[\, D_1(w; u, v) + D_2(w; u, v) + D_3(w; u, v) \,\big], \qquad (4.3.34)
$$

where

$$
\begin{aligned}
D_1(w;u,v) &= \sum_{E\in\mathcal{T}_h} \delta_E \big[\, (\,\boldsymbol{\beta}(u)\cdot\nabla w,\ \ \boldsymbol{\beta}(u)\cdot\nabla v\,)_E + (\,w\,\partial_u\boldsymbol{\beta}(u)\cdot\nabla u,\ \ \boldsymbol{\beta}(u)\cdot\nabla v\,)_E \\
&\qquad\qquad + (\,\boldsymbol{\beta}(u)\cdot\nabla u,\ \ w\,\partial_u\boldsymbol{\beta}(u)\cdot\nabla v\,)_E\,\big], \qquad (4.3.35)\\
D_2(w;u,v) &= \sum_{E\in\mathcal{T}_h} (\,\boldsymbol{\beta}(u)\cdot\nabla w + \ w\,\partial_u r(u) + \ w\,\partial_u\boldsymbol{\beta}(u)\cdot\nabla u,\ \ v\,)_E\,, \quad \text{and} \qquad (4.3.36)\\
D_3(w;u,v) &= \sum_{E\in\mathcal{T}_h} \delta_E \big[\, \big(-\nabla\cdot\epsilon\,\boldsymbol{\Pi}^0_{p-1}\nabla u + \ r(u),\ \ w\,\partial_u\boldsymbol{\beta}(u)\cdot\nabla v\,\big)_E \\
&\qquad\qquad + \big(-\nabla\cdot\epsilon\,\boldsymbol{\Pi}^0_{p-1}\nabla w + \ w\,\partial_u r(u),\ \ \boldsymbol{\beta}(u)\cdot\nabla v\,\big)_E\,\big]. \qquad (4.3.37)
\end{aligned}
$$

Next, for every $u\in V_h^p$, we define a bounded linear operator $\widehat{D}M_h(u) : I\times V_h^p \to Y_h$, such that for any $w\in V_h^p$,

$$
\langle \widehat{D}M_h(\mu,\ u)w,\ v\rangle := \mu\,\big[\,\widehat{D}_1(w;u,v) + \widehat{D}_2(w;u,v) + \widehat{D}_3(w;u,v)\,\big], \qquad (4.3.38)
$$

where,

$$
\begin{aligned}
\widehat{D}_1(w;u,v) &= \sum_{E\in\mathcal{T}_h} \delta_E \big[\, \big(\,\boldsymbol{\beta}(\Pi^0_p u)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla w + \ \Pi^0_p w\,\partial_u\boldsymbol{\beta}(\Pi^0_p u)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u,\ \boldsymbol{\beta}(\Pi^0_p u)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v\,\big)_E \\
&\qquad + \big(\,\boldsymbol{\beta}(\Pi^0_p u)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u,\ \Pi^0_p w\,\partial_u\boldsymbol{\beta}(\Pi^0_p u)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v\,\big)_E \\
&\qquad + \tilde{\mathcal{S}}^2\, S^E((I-\Pi^\nabla_p)w,\ (I-\Pi^\nabla_p)v)\,\big], \qquad (4.3.39)\\
\widehat{D}_2(w;u,v) &= \sum_{E\in\mathcal{T}_h} \big[\, \big(\,\boldsymbol{\beta}(\Pi^0_p u)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla w,\ \Pi^0_p v\,\big)_E + \big(\,\Pi^0_p w\,\partial_u r(\Pi^0_p u),\ \Pi^0_p v\,\big)_E \\
&\qquad + \big(\,\Pi^0_p w\,\partial_u\boldsymbol{\beta}(\Pi^0_p u)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u,\ \Pi^0_p v\,\big)_E\,\big], \qquad \text{and} \qquad (4.3.40)\\
\widehat{D}_3(w;u,v) &= \sum_{E\in\mathcal{T}_h} \delta_E \big[\, \big(-\nabla\cdot\epsilon\,\boldsymbol{\Pi}^0_{p-1}\nabla u + \ r(\Pi^0_p u),\ \Pi^0_p w\,\partial_u\boldsymbol{\beta}(\Pi^0_p u)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v\,\big)_E \\
&\qquad + \big(-\nabla\cdot\epsilon\,\boldsymbol{\Pi}^0_{p-1}\nabla w + \ \Pi^0_p w\,\partial_u r(\Pi^0_p u),\ \boldsymbol{\beta}(\Pi^0_p u)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v\,\big)_E\,\big]. \qquad (4.3.41)
\end{aligned}
$$

Lemma 4.6. *Consider* (4.1.2)*, Assumptions 4.2-4.4(section 1.3),* (4.2.12) *and* $(\mu,u_\mu)\in\mathcal{B}$*. Let* $\nabla\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u_\mu|_E \in L^\infty(E),\ \forall E\in\mathcal{T}_h$*, then the following estimate is attained.*

$$
\|\widehat{D}M_h(\mu,\ u_{I,\mu}) - DM(\mu,u_\mu)\|_{L(V_h^p,Y_h)} \le \mathcal{C}_4\,h, \qquad (4.3.42)
$$

where $\|\cdot\|_{L(V_h,Y_h)} = \sup\limits_{0\neq z_h\in V_h} \dfrac{\|\cdot\|_*}{|z_h|_{1,\Omega}}$.

Proof. As earlier, we denote $u := u_\mu$ and $u_I := u_{I,\mu}$. The following estimate will be used in the proof. Using triangle inequality, (4.3.2) and (4.3.4), we have,

$$\begin{aligned}
\|\Pi_p^0 u_I - u\|_{1,\Omega} &= \sum_{E\in\mathcal{T}_h} \|\Pi_p^0 u_I - u\|_{1,E} \\
&\leq \sum_{E\in\mathcal{T}_h} \|\Pi_p^0 u_I - \Pi_p^0 u\|_{1,E} + \sum_{E\in\mathcal{T}_h} \|\Pi_p^0 u - u\|_{1,E} \\
&\leq \sum_{E\in\mathcal{T}_h} \|u_I - u\|_{1,E} + \sum_{E\in\mathcal{T}_h} C\, h_E^s\, |u|_{1+s,E} \\
&\leq \|u_I - u\|_{1,\Omega} + C\, h^s\, |u|_{1+s,\Omega} \leq C\, h^s\, |u|_{1+s,\Omega}. \qquad (4.3.43)
\end{aligned}$$

We estimate the required terms one by one.

Consider $0 \neq w_h, v_h \in V_h$. Let $\tau_1 := \mu\,[\widehat{D}_1(w_h; u_I, v_h) - D_1(w_h; u, v_h)]$. Then,

$$\begin{aligned}
\tau_1 &= \sum_{E\in\mathcal{T}_h} \Big\{ \delta_E \,\big[\, \big(\boldsymbol{\beta}(\Pi_p^0 u_I)\cdot \mathbf{\Pi}_{p-1}^0 \nabla w_h,\ \boldsymbol{\beta}(\Pi_p^0 u_I)\cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E \\
&\qquad\qquad - \big(\, \boldsymbol{\beta}(u)\cdot\nabla w_h,\ \boldsymbol{\beta}(u)\cdot \nabla v_h \,\big)_E \,\big] \\
&\quad + \delta_E \,\big[\, \big(\Pi_p^0 w_h\, \partial_u \boldsymbol{\beta}(\Pi_p^0 u_I)\cdot \mathbf{\Pi}_{p-1}^0 \nabla u_I,\ \boldsymbol{\beta}(\Pi_p^0 u_I)\cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E \\
&\qquad\qquad - (w_h\, \partial_u \boldsymbol{\beta}(u)\cdot \nabla u,\ \boldsymbol{\beta}(u)\cdot\nabla v_h)_E \,\big] \\
&\quad + \delta_E \,\big[\, \big(\boldsymbol{\beta}(\Pi_p^0 u_I)\cdot \mathbf{\Pi}_{p-1}^0 \nabla u_I,\ \Pi_p^0 w_h\, \partial_u \boldsymbol{\beta}(\Pi_p^0 u_I)\cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E \\
&\qquad\qquad - (\boldsymbol{\beta}(u)\cdot\nabla u,\ w_h\, \partial_u \boldsymbol{\beta}(u)\cdot \nabla v_h)_E \,\big] \\
&\quad + \delta_E\, \tilde{\mathbb{S}}^2\, S^E((I - \Pi_p^\nabla) w_h,\ (I-\Pi_p^\nabla) v_h) \Big\} \\
&= \tau_{11} + \tau_{12} + \tau_{13} + \tau_{14}. \qquad (4.3.44)
\end{aligned}$$

To τ_{11}, we add and subtract the term

$$\sum_{E\in\mathcal{T}_h} \delta_E\, \big([\,\boldsymbol{\beta}(\Pi_p^0 u_I) + \boldsymbol{\beta}(u)\,]\cdot \mathbf{\Pi}_{p-1}^0 \nabla w_h + \boldsymbol{\beta}(u)\cdot\nabla w_h,\ \boldsymbol{\beta}(u)\cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E .$$

Then, we get,

$$\begin{aligned}
\tau_{11} &= \sum_{E\in\mathcal{T}_h} \Big\{ \delta_E\, \big(\boldsymbol{\beta}(\Pi_p^0 u_I)\cdot \mathbf{\Pi}_{p-1}^0 \nabla w_h,\ [\,\boldsymbol{\beta}(\Pi_p^0 u_I) - \boldsymbol{\beta}(u)\,]\cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E \\
&\quad + \delta_E\, \big([\,\boldsymbol{\beta}(\Pi_p^0 u_I) - \boldsymbol{\beta}(u)\,]\cdot \mathbf{\Pi}_{p-1}^0 \nabla w_h,\ \boldsymbol{\beta}(u)\cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E \\
&\quad + \delta_E\, \big(\boldsymbol{\beta}(u)\cdot [\mathbf{\Pi}_{p-1}^0 \nabla w_h - \nabla w_h],\ \boldsymbol{\beta}(u)\cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E \\
&\quad + \delta_E\, \big(\boldsymbol{\beta}(u)\cdot\nabla w_h,\ \boldsymbol{\beta}(u)\cdot [\mathbf{\Pi}_{p-1}^0 \nabla v_h - \nabla v_h] \big)_E \Big\}.
\end{aligned}$$

Using Remark 4.5, (4.1.2), generalised Hölder's inequality, (4.3.6), (4.3.43) and (4.3.3), we get

$$
\begin{aligned}
\tau_{11} \;\; &\leq \;\; [\mathcal{Q}(C\lambda)]^2 \sum_{E\in\mathcal{T}_h} \delta_E \left\{ 2\, \|\mathbf{\Pi}^0_{p-1}\nabla w_h\|_E \, \|\Pi^0_p u_I - u\|_{L^3(E)} \, \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{L^6(E)} \right.\\
&\qquad \left. + \|\mathbf{\Pi}^0_{p-1}\nabla w_h - \nabla w_h\|_E \, \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_E \, + \, \|\mathbf{\Pi}^0_{p-1}\nabla w_h\|_E \, \|\mathbf{\Pi}^0_{p-1}\nabla v_h - \nabla v_h\|_E \right\}\\
&\leq \;\; C\, [\mathcal{Q}(C\lambda)]^2 \sum_{E\in\mathcal{T}_h} \delta_E \left\{ 2\, |w_h|_{1,E} \, \|\Pi^0_p u_I - u\|_{1,E} \, h_E^{-1} |v_h|_{1,E} \right.\\
&\qquad \left. + |\Pi^{\nabla}_k w_h - w_h|_{1,E} \, |v_h|_{1,E} \, + \, |w_h|_{1,E} \, |\Pi^{\nabla}_k v_h - v_h|_{1,E} \right\} \quad (\text{ use (2.6,[82]), (5.22,[12]) })\\
&\leq \;\; C\, [\mathcal{Q}(C\lambda)]^2 \sum_{E\in\mathcal{T}_h} 2\, \delta_E \left\{ |w_h|_{1,E} \, \|\Pi^0_p u_I - u\|_{1,E} \, h_E^{-1} |v_h|_{1,E} + C\, |w_h|_{1,E} \, |v_h|_{1,E} \right\}\\
&\leq \;\; C\, [\mathcal{Q}(C\lambda)]^2 \, h \left[C\, h^s \, |u|_{1+s,\Omega} + 1 \right] |w_h|_{1,\Omega} \, |v_h|_{1,\Omega}. \quad (\text{ use (4.2.12) }) \qquad (4.3.45)
\end{aligned}
$$

Let us define,

$$
\begin{aligned}
t_{1*} := \sum_{E\in\mathcal{T}_h} \delta_E \Big[& \left(\Pi^0_p w_h \, \partial_u \boldsymbol{\beta}(u) \cdot \mathbf{\Pi}^0_{p-1}\nabla u_I, \, \boldsymbol{\beta}(\Pi^0_p u_I) \cdot \mathbf{\Pi}^0_{p-1}\nabla v_h \right)_E \\
& + \left(\Pi^0_p w_h \, \partial_u \boldsymbol{\beta}(u) \cdot \mathbf{\Pi}^0_{p-1}\nabla u_I, \, \boldsymbol{\beta}(u) \cdot \mathbf{\Pi}^0_{p-1}\nabla v_h \right)_E \\
& + \left(\Pi^0_p w_h \, \partial_u \boldsymbol{\beta}(u) \cdot \nabla u + w_h \, \partial_u \boldsymbol{\beta}(u) \cdot \nabla u, \;\; \boldsymbol{\beta}(u) \cdot \mathbf{\Pi}^0_{p-1}\nabla v_h \right)_E \Big].
\end{aligned}
$$

Adding and subtracting t_{1*} to τ_{12}, we get,

$$
\begin{aligned}
\tau_{12} \;\; := \;\; \sum_{E\in\mathcal{T}_h} \delta_E \Big[& \left(\Pi^0_p w_h \left[\partial_u \boldsymbol{\beta}(\Pi^0_p u_I) - \partial_u \boldsymbol{\beta}(u) \right] \cdot \mathbf{\Pi}^0_{p-1}\nabla u_I, \, \boldsymbol{\beta}(\Pi^0_p u_I) \cdot \mathbf{\Pi}^0_{p-1}\nabla v_h \right)_E \\
& + \left(\Pi^0_p w_h \, \partial_u \boldsymbol{\beta}(u) \cdot \mathbf{\Pi}^0_{p-1}\nabla u_I, \, \left[\boldsymbol{\beta}(\Pi^0_p u_I) - \boldsymbol{\beta}(u) \right] \cdot \mathbf{\Pi}^0_{p-1}\nabla v_h \right)_E \\
& + \left(\Pi^0_p w_h \, \partial_u \boldsymbol{\beta}(u) \cdot \left[\mathbf{\Pi}^0_{p-1}\nabla u_I - \nabla u \right] + \left[\Pi^0_p w_h - w_h \right] \partial_u \boldsymbol{\beta}(u) \cdot \nabla u, \;\; \boldsymbol{\beta}(u) \cdot \mathbf{\Pi}^0_{p-1}\nabla v_h \right)_E \\
& + \left(w_h \, \partial_u \boldsymbol{\beta}(u) \cdot \nabla u, \;\; \boldsymbol{\beta}(u) \cdot \left[\mathbf{\Pi}^0_{p-1}\nabla v_h - \nabla v_h \right] \right)_E \Big] \qquad (4.3.46)
\end{aligned}
$$

Now estimating τ_{12}, we obtain,

$$
\begin{aligned}
\tau_{12} \;\; &\leq \;\; \sum_{E\in\mathcal{T}_h} [\mathcal{Q}(C\lambda)]^2 \, \delta_E \left\{ 2\, \|\Pi^0_p w_h\|_{L^6(E)} \, \|\Pi^0_p u_I - u\|_{L^6(E)} \, \|\mathbf{\Pi}^0_{p-1}\nabla u_I\|_{L^6(E)} \, \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_E \right.\\
&\qquad + \|\Pi^0_p w_h\|_{L^6(E)} \, \|\mathbf{\Pi}^0_{p-1}\nabla u_I - \nabla u\|_E \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{L^3(E)} \\
&\qquad + \|\Pi^0_p w_h - w_h\|_{L^6(E)} \, \|\nabla u\|_{L^3(E)} \, \|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_E \\
&\qquad \left. + \|w_h\|_{L^6(E)} \, \|\nabla u\|_{L^3(E)} \, \|\Pi^0_{k-1}\nabla v_h - \nabla v_h\|_E \right\}\\
&\leq \;\; \sum_{E\in\mathcal{T}_h} [\mathcal{Q}(C\lambda)]^2 \, \delta_E \left\{ 2\, \|w_h\|_{1,E} \, \|\Pi^0_p u_I - u\|_{1,E} \, h_E^{-1} \, \|\nabla u_I\|_E \, |v_h|_{1,E} \right.\\
&\qquad + \left[\|w_h\|_{1,E} \, \|\mathbf{\Pi}^0_{p-1}\nabla u_I - \nabla u\|_E \, h_E^{-1} + \|\Pi^0_p w_h - w_h\|_{1,E} \, \|\nabla u\|_{1,E} \right] |v_h|_{1,E} \\
&\qquad \left. + \|w_h\|_{1,E} \, \|\nabla u\|_{1,E} \, |\Pi^{\nabla}_k v_h - v_h|_{1,E} \right\} \quad (\text{ use (4.3.6), (2.6,[82]) }).
\end{aligned}
$$

Using Hölder's inequality, (4.2.12), $|u_I|_{1,\Omega} \leq C|u|_{1,\Omega}$, (4.3.2), (4.3.3), Poincaré inequality and (4.3.43), we get,

$$\tau_{12} \;\; \leq \;\; C\,[\mathcal{Q}(C\lambda)]^2\,h\,\big[\,|u|_{1,\Omega}\,h^{s-1}\,|u|_{1+s,\Omega} + 1 + 2\,|u|_{2,\Omega}\,\big]\,|w_h|_{1,\Omega}\,|v_h|_{1,\Omega}. \quad (4.3.47)$$

Same result holds for τ_{13}, i.e.

$$\tau_{13} \;\; \leq \;\; C\,[\mathcal{Q}(C\lambda)]^2\,h\,\big[\,|u|_{1,\Omega}\,h^{s-1}\,|u|_{1+s,\Omega} + 1 + 2\,|u|_{2,\Omega}\,\big]\,|w_h|_{1,\Omega}\,|v_h|_{1,\Omega}. \quad (4.3.48)$$

Next, using (4.2.5), (4.2.11), (4.2.12) and (4.3.3), we get,

$$\tau_{14} \;\; \leq \;\; C\,[\mathcal{Q}(C\lambda)]^2\,h\,|w_h|_{1,\Omega}\,|v_h|_{1,\Omega}. \quad (4.3.49)$$

Substituting (4.3.45), (4.3.47), (4.3.48) and (4.3.49) into the equation (4.3.44), we get,

$$\tau_1 \leq C\,h\,\mathcal{C}_{41}\,|w_h|_{1,\Omega}\,|v_h|_{1,\Omega}, \quad (4.3.50)$$

where $\mathcal{C}_{41} := [\mathcal{Q}(C\lambda)]^2\,\big[\,4 + 2|u|_{2,\Omega} + h^{s-1}\,|u|_{s+1,\Omega}\,(\,h + 4|u|_{1,\Omega}\,)\,\big]$.
Let $\tau_2 := \mu\,[\widehat{D}_2(w_h; u_I, v_h) - D_2(w_h; u, v_h)]$. Then elaborating, we have

$$\begin{aligned}
\tau_2 \;\; := \;\; & \sum_{E\in\mathcal{T}_h} \Big\{ \big[\, \big(\boldsymbol{\beta}(\Pi_p^0 u_I)\cdot \boldsymbol{\Pi}_{p-1}^0 \nabla w_h,\; \Pi_p^0 v_h\big)_E - \big(\boldsymbol{\beta}(u)\cdot\nabla w_h,\; v_h\big)_E \,\big] \\
& + \big[\, \big(\Pi_p^0 w_h\,\partial_u r(\Pi_p^0 u_I),\; \Pi_p^0 v_h\big)_E - \big(w_h\,\partial_u r(u),\; v_h\big)_E \,\big] \\
& + \big[\, \big(\Pi_p^0 w_h\,\partial_u \boldsymbol{\beta}(\Pi_p^0 u_I)\cdot \boldsymbol{\Pi}_{p-1}^0 \nabla u_I,\; \Pi_p^0 v_h\big)_E - \big(w_h\,\partial_u\boldsymbol{\beta}(u)\cdot\nabla u,\; v_h\big)_E \,\big] \Big\} \\
= \;\; & \tau_{21} + \tau_{22} + \tau_{23}. \qquad (4.3.51)
\end{aligned}$$

To τ_{21}, we add and subtract $\sum_{E\in\mathcal{T}} \big(\boldsymbol{\beta}(u)\cdot\boldsymbol{\Pi}_{p-1}^0\nabla w_h,\; \Pi_p^0 v_h + v_h\big)_E$ obtaining,

$$\begin{aligned}
\tau_{21} \;\; = \;\; & \sum_{E\in\mathcal{T}} \big[\, \big(\,[\,\boldsymbol{\beta}(\Pi_p^0 u_I) - \boldsymbol{\beta}(u)\,]\cdot\boldsymbol{\Pi}_{p-1}^0\nabla w_h,\; \Pi_p^0 v_h\big)_E + \big(\boldsymbol{\beta}(u)\cdot\boldsymbol{\Pi}_{p-1}^0\nabla w_h,\; \Pi_p^0 v_h - v_h\big)_E \\
& \qquad + \big(\boldsymbol{\beta}(u)\cdot[\,\boldsymbol{\Pi}_{p-1}^0\nabla w_h - \nabla w_h\,],\; v_h\big)_E \,\big] \\
= \;\; & \sum_{E\in\mathcal{T}} \big[\, \big(\,[\,\boldsymbol{\beta}(\Pi_p^0 u_I) - \boldsymbol{\beta}(u)\,]\cdot\boldsymbol{\Pi}_{p-1}^0\nabla w_h,\; \Pi_p^0 v_h\big)_E + \big(\boldsymbol{\beta}(u)\cdot\boldsymbol{\Pi}_{p-1}^0\nabla w_h,\; \Pi_p^0 v_h - v_h\big)_E \\
& \qquad + \big(\boldsymbol{\beta}(u)\cdot\nabla w_h,\; v_h - \Pi_{k-1}^0 v_h\big)_E \,\big] \\
\leq \;\; & \sum_{E\in\mathcal{T}_h} \mathcal{Q}(C\lambda)\,\big[\,\|\Pi_p^0 u_I - u\|_{L^3(E)}\,\|\boldsymbol{\Pi}_{p-1}^0\nabla w_h\|_E\,\|\Pi_p^0 v_h\|_{L^6(E)} \\
& \qquad + \|\boldsymbol{\Pi}_{p-1}^0\nabla w_h\|_E\,\|\Pi_p^0 v_h - v_h\|_E + \|\nabla w_h\|_E\,\|v_h - \Pi_{k-1}^0 v_h\|_E\,\big].
\end{aligned}$$

Then, we get

$$
\begin{aligned}
\tau_{21} &\le \sum_{E\in\mathcal{T}_h} \mathcal{Q}(C\lambda)\left[\,\|\Pi_p^0 u_I - u\|_{1,E}\,|w_h|_{1,E}\,\|\Pi_p^0 v_h\|_{1,E}\right.\\
&\qquad \left. + |w_h|_{1,E}\,C\,h_E\,|v_h|_{1,E} + |w_h|_{1,E}\,C\,h_E\,|v_h|_{1,E}\,\right] \quad \text{(use (4.3.6), (4.3.2))}\\
&\le C\,\mathcal{Q}(C\lambda)\left[\,\|\Pi_p^0 u_I - u\|_{1,\Omega} + h\,\right]|w_h|_{1,\Omega}\,|v_h|_{1,\Omega} \quad \text{(use Hölder's inequality)}\\
&\le C\,\mathcal{Q}(C\lambda)\,h\left[\,h^{s-1}\,|u|_{s+1,\Omega} + 1\,\right]|w_h|_{1,\Omega}\,|v_h|_{1,\Omega}. \quad \text{((4.3.43))} \qquad (4.3.52)
\end{aligned}
$$

To τ_{22}, we add and subtract $\sum_{E\in\mathcal{T}_h}\big(\,w_h\,\partial_u r(\Pi_p^0 u_I),\ \Pi_p^0 v_h + v_h\,\big)_E$ obtaining,

$$
\begin{aligned}
\tau_{22} &:= \sum_{E\in\mathcal{T}_h}\Big[\,\big(\,[\,\Pi_p^0 w_h - w_h\,]\,\partial_u r(\Pi_p^0 u_I),\ \Pi_p^0 v_h\,\big)_E + \big(\,w_h\,\partial_u r(\Pi_p^0 u_I),\ \Pi_p^0 v_h - v_h\,\big)_E\\
&\qquad + \big(\,w_h\,[\,\partial_u r(\Pi_p^0 u_I) - \partial_u r(u)\,],\ v_h\,\big)_E\,\Big].\\
&\le \sum_{E\in\mathcal{T}_h}\mathcal{Q}(C\lambda)\left[\,\|\Pi_p^0 w_h - w_h\|_E\,\|\Pi_p^0 v_h\|_E + \|\Pi_p^0 w_h\|_E\,\|\Pi_p^0 v_h - v_h\|_E\right.\\
&\qquad \left. + \|w_h\|_{L^3(E)}\,\|\Pi_p^0 u_I - u\|_{L^6(E)}\,\|v_h\|_E\,\right]\\
&\le C\sum_{E\in\mathcal{T}_h}\mathcal{Q}(C\lambda)\left[\,2\,h_E\,|w_h|_{1,E}\,|v_h|_{1,E} + \|\Pi_p^0 u_I - u\|_{1,E}\,\|w_h\|_{1,E}\,\|v_h\|_{1,E}\,\right]\\
&\le C\,\mathcal{Q}(C\lambda)\,h\,[\,2 + h^{s-1}\,|u|_{1+s,\Omega}\,]\,|w_h|_{1,E}\,|v_h|_{1,E}. \qquad (4.3.53)
\end{aligned}
$$

To τ_{23}, we add and subtract the term
$\sum_{E\in\mathcal{T}_h}\Big[\,\big(\,w_h\,\partial_u\boldsymbol{\beta}(\Pi_p^0 u_I)\cdot\boldsymbol{\Pi}_{p-1}^0\nabla u_I,\ \Pi_p^0 v_h\,\big)_E + \big(\,w_h\,\partial_u\boldsymbol{\beta}(\Pi_p^0 u_I)\cdot\nabla u,\ \Pi_p^0 v_h + v_h\,\big)_E\,\Big].$

$$
\begin{aligned}
\tau_{23} &= \sum_{E\in\mathcal{T}_h}\Big[\,\big(\,[\,\Pi_p^0 w_h - w_h\,]\,\partial_u\boldsymbol{\beta}(\Pi_p^0 u_I)\cdot\boldsymbol{\Pi}_{p-1}^0\nabla u_I + w_h\ \partial_u\boldsymbol{\beta}(\Pi_p^0 u_I)\cdot[\,\boldsymbol{\Pi}_{p-1}^0\nabla u_I - \nabla u\,],\ \Pi_p^0 v_h\,\big)_E\\
&\qquad + \big(\,w_h\,\partial_u\boldsymbol{\beta}(\Pi_p^0 u_I)\cdot\nabla u,\ \Pi_p^0 v_h - v_h\,\big)_E + \big(\,w_h\,[\,\partial_u\boldsymbol{\beta}(\Pi_p^0 u_I) - \partial_u\boldsymbol{\beta}(u)\,]\cdot\nabla u,\ v_h\,\big)_E\,\Big]\\
&\le \sum_{E\in\mathcal{T}_h}\mathcal{Q}(C\lambda)\Big\{\,[\,\|(\Pi_p^0 - I)w_h\|_{6,E}\,\|\boldsymbol{\Pi}_{p-1}^0\nabla u_I\|_E + \|w_h\|_{6,E}\,\|\boldsymbol{\Pi}_{p-1}^0\nabla u_I - \nabla u\|_E\,]\,\|\Pi_p^0 v_h\|_{3,E}\\
&\qquad + \|w_h\|_{L^3(E)}\,\|\nabla u\|_{6,E}\,\|\Pi_p^0 v_h - v_h\|_E + \|w_h\|_{6,E}\,\|\Pi_p^0 u_I - u\|_{6,E}\,\|\nabla u\|_E\,\|v_h\|_{6,E}\,\Big\}
\end{aligned}
$$

Simplifying further, we have

$$
\begin{aligned}
\tau_{23} &\le C\sum_{E\in\mathcal{T}_h}\mathcal{Q}(C\lambda)\Big\{\,[\,h_E\,|w_h|_{1,E}\,2\,|u|_{1,E} + \|w_h\|_{1,E}\,\|\boldsymbol{\Pi}_{p-1}^0\nabla u_I - \nabla u\|_E\,]\,\|v_h\|_{1,E}\\
&\qquad + [\,h_E\,|u|_{2,E} + \|\Pi_p^0 u_I - u\|_{1,E}\,|u|_{1,E}\,]\,\|w_h\|_{1,E}\,\|v_h\|_{1,E}\,\Big\}\\
&\le C\,\mathcal{Q}(C\lambda)\,h\,\big\{\,2\,|u|_{1,\Omega} + h^{s-1}\,|u|_{s+1,\Omega} + |u|_{2,\Omega} + h^{s-1}\,|u|_{s+1,\Omega}\,|u|_{1,\Omega}\,\big\}\,|w_h|_{1,\Omega}\,|v_h|_{1,\Omega}.
\end{aligned}
$$

(4.3.54)

Substituting (4.3.52), (4.3.53) and (4.3.54) into the equation (4.3.51), we get,

$$\tau_2 \leq C\, h\, \mathcal{C}_{42}\, |w_h|_{1,\Omega}\, |v_h|_{1,\Omega}, \tag{4.3.55}$$

where $\quad \mathcal{C}_{42} := \mathcal{Q}(C\lambda)\left[\, 3 + |u|_{2,\Omega} + 2|u|_{1,\Omega} + h^{s-1}\, |u|_{s+1,\Omega}\, (\, 3 + |u|_{1,\Omega}\,)\,\right].$
Let $\tau_3 := \mu\,[\widehat{D}_3(w_h; u_I, v_h) - D_3(w_h; u, v_h)]$. Then, we obtain,

$$\begin{aligned}
\tau_3 \quad := \quad & \sum_{E\in\mathcal{T}_h} \Big\{\, \delta_E \,\big[\, \big(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla u_I,\ \Pi^0_p w_h\, \partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E \\
& \qquad\qquad - \big(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla u,\ w_h\,\partial_u\boldsymbol{\beta}(u)\cdot\nabla v_h\,\big)_E\,\big] \\
& \quad + \delta_E\,\big[\,\big(\,r(\Pi^0_p u_I),\ \Pi^0_p w_h\,\partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E \\
& \qquad - (\,r(u),\ w_h\,\partial_u\boldsymbol{\beta}(u)\cdot\nabla v_h\,)_E\,\big] \\
& \quad + \delta_E\,\big[\,\big(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla w_h,\ \boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E \\
& \qquad - \big(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla w_h,\ \boldsymbol{\beta}(u)\cdot\nabla v_h\,\big)_E\,\big] \\
& \quad + \delta_E\,\big[\,\big(\,\Pi^0_p w_h\,\partial_u r(\Pi^0_p u_I),\ \boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E \\
& \qquad - (\,w_h\,\partial_u r(u),\ \boldsymbol{\beta}(u)\cdot\nabla v_h\,)_E\,\big]\Big\} \\
= \quad & \tau_{31} + \tau_{32} + \tau_{33} + \tau_{34}.
\end{aligned} \tag{4.3.56}$$

To τ_{31}, we add and subtract

$$\begin{aligned}
& \sum_{E\in\mathcal{T}_h} \delta_E\,\big(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla u,\ [\,\Pi^0_p w_h + w_h\,]\,\partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E\,, \\
& \text{and} \quad \sum_{E\in\mathcal{T}_h} \delta_E\,\big(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla u,\ w_h\,\partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\nabla v_h\,\big)_E
\end{aligned}$$

obtaining,

$$\begin{aligned}
\tau_{31} \quad = \quad & \sum_{E\in\mathcal{T}_h} \delta_E\,\big[\,\big(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}[\,\nabla u_I - \nabla u\,],\ \Pi^0_p w_h\,\partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E \\
& \quad + \big(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla u,\ [\,\Pi^0_p w_h - w_h\,]\,\partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\,\big)_E \\
& \quad + \big(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla u,\ w_h\,\partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot[\,\mathbf{\Pi}^0_{p-1}\nabla v_h - \nabla v_h\,]\big)_E \\
& \quad + \big(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla u,\ w_h\,[\,\partial_u\boldsymbol{\beta}(\Pi^0_p u_I) - \partial_u\boldsymbol{\beta}(u)\,]\cdot\nabla v_h\,\big)_E\,\big].
\end{aligned}$$

Using the assumption $\nabla\cdot\mathbf{\Pi}^0_{p-1}\nabla u|_E \in L^\infty(E),\ \forall E \in \mathcal{T}_h$, MVT, (4.1.2), generalised Hölder's inequality(with (4.3.6), we get,

$$\begin{aligned}
\tau_{31} \;\leq\;& \mathcal{Q}(C\lambda) \sum_{E\in\mathcal{T}_h} \delta_E \left[\, \| -\nabla\cdot\epsilon\boldsymbol{\Pi}^0_{p-1}(\nabla u_I - \nabla u)\|_E \, \|\Pi^0_p w_h\|_{L^3(E)} \, \|\boldsymbol{\Pi}^0_{p-1}\nabla v_h\|_{L^6(E)} \right.\\
&\quad + \epsilon\, \|\nabla\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u\|_{\infty,E} \, \|\Pi^0_p w_h - w_h\|_E \, \|\boldsymbol{\Pi}^0_{p-1}\nabla v_h\|_E \\
&\quad + \epsilon\, \|\nabla\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u\|_{\infty,E} \, \|w_h\|_E \, \|\boldsymbol{\Pi}^0_{p-1}\nabla v_h - \nabla v_h\|_E \\
&\quad \left. + \epsilon\, \|\nabla\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u\|_{\infty,E} \, \|w_h\|_{L^3(E)} \, \|\Pi^0_p u_I - u\|_{L^6(E)} \, \|\nabla v_h\|_E \,\right] \\
\leq\;& \mathcal{Q}(C\lambda) \sum_{E\in\mathcal{T}_h} \delta_E \,\epsilon\, C \left[\, h_E^{-1} \|\boldsymbol{\Pi}^0_{p-1}(\nabla u_I - \nabla u)\|_E \, \|\Pi^0_p w_h\|_{1,E} \, \|\boldsymbol{\Pi}^0_{p-1}\nabla v_h\|_{1,E} \right. \text{ (use (4.3.1))}\\
&\quad + \|\nabla\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u\|_{L^\infty(E)} \, h_E \, |w_h|_{1,E} \, |v_h|_{1,E} + \|\nabla\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u\|_{\infty,E} \, |w_h|_{1,E} \, |\Pi^\nabla_p v_h - v_h|_{1,E} \\
&\quad \left. + \|\nabla\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u\|_{\infty,E} \, \|w_h\|_{1,E} \, \|\Pi^0_p u_I - u\|_{1,E} \, |v_h|_{1,E} \,\right] \\
\leq\;& \mathcal{Q}(C\lambda)\, h^2\, C \left[\, h^{s-2} \, |u|_{1+s,\Omega} \, |w_h|_{1,\Omega} \, |v_h|_{1,\Omega} \right. \text{ (use (2.6,[82]), Hölder's \& (4.3.4))}\\
&\quad + \|\nabla\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u\|_{\infty,E} \, (h+1) \, |w_h|_{1,\Omega} \, |v_h|_{1,\Omega} \\
&\quad \left. + \|\nabla\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u\|_{\infty,E} \, \|w_h\|_{1,\Omega} \, h^s \, |u|_{s+1,\Omega} \, |v_h|_{1,\Omega} \,\right] \text{ (use (4.3.43))}\\
\leq\;& C\,\mathcal{Q}(C\lambda)\, h \left[\, h^{s-1}\, |u|_{s+1,\Omega} + (h^2 + h + h^{s+1}\, |u|_{s+1,\Omega}) \, \|\nabla\cdot\boldsymbol{\Pi}^0_{p-1}\nabla u\|_{\infty,E} \,\right]. \qquad (4.3.57)
\end{aligned}$$

To τ_{32}, we add and subtract the terms $\sum_{E\in\mathcal{T}_h} \delta_E \left\{ \left(r(u),\, \Pi^0_p w_h \, \partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h \right)_E + \right.$ $\left.\left(r(u),\, w_h\, \partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot(\boldsymbol{\Pi}^0_{p-1}\nabla v_h + \nabla v_h) \right)_E \right\}$ and to get

$$\begin{aligned}
\tau_{32} \;=\;& \sum_{E\in\mathcal{T}_h} \delta_E \left\{ \left(r(\Pi^0_p u_I) - r(u),\; \Pi^0_p w_h \, \partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h \right)_E \right.\\
&\quad + \left(r(u),\; (\Pi^0_p w_h - w_h)\, \partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\boldsymbol{\Pi}^0_{p-1}\nabla v_h \right)_E \\
&\quad + \left(r(u),\; w_h\, \partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot(\boldsymbol{\Pi}^0_{p-1}\nabla v_h - \nabla v_h) \right)_E \\
&\quad \left. + \left(r(u),\; w_h\, (\partial_u\boldsymbol{\beta}(\Pi^0_p u_I) - \partial_u\boldsymbol{\beta}(u))\cdot\nabla v_h \right)_E \right\}.
\end{aligned}$$

Using the MVT, (4.1.2), generalised Hölder's inequality, (4.2.12) and (4.3.6), we get,

$$\begin{aligned}
\tau_{32} \;\leq\;& [\mathcal{Q}(C\lambda)]^2 \sum_{E\in\mathcal{T}_h} \delta_E \left\{ \|\Pi^0_p u_I - u\|_{L^6(E)} \, \|\Pi^0_p w_h\|_{L^3(E)} \, \|\boldsymbol{\Pi}^0_{p-1}\nabla v_h\|_E \right.\\
&\quad + \|\Pi^0_p w_h - w_h\|_E \, \|\boldsymbol{\Pi}^0_{p-1}\nabla v_h\|_E + \|w_h\|_E \, \|\boldsymbol{\Pi}^0_{p-1}\nabla v_h - \nabla v_h\|_E \\
&\quad \left. + \|\Pi^0_p u_I - u\|_{L^6(E)} \, \|w_h\|_{L^3(E)} \, \|\nabla v_h\|_E \right\} \\
\leq\;& C\,[\mathcal{Q}(C\lambda)]^2 \sum_{E\in\mathcal{T}_h} \delta_E \left\{ \|\Pi^0_p u_I - u\|_{1,E} \, |w_h|_{1,E} \, |v_h|_{1,E} + h_E \, |w_h|_{1,E} \, |v_h|_{1,E} \right.\\
&\quad \left. + \|w_h\|_E \, \|\Pi^\nabla_p v_h - v_h\|_E + \|\Pi^0_p u_I - u\|_{1,E} \, \|w_h\|_{1,E} \, \|v_h\|_{1,E} \right\} \\
\leq\;& C\,[\mathcal{Q}(C\lambda)]^2\, h \left[\, 2\, h^s \, |u|_{1+s,\Omega} + h + 1 \,\right] |w_h|_{1,\Omega} \, |v_h|_{1,\Omega}. \qquad (4.3.58)
\end{aligned}$$

To τ_{33}, we add and subtract $\sum_{E\in\mathcal{T}_h} \delta_E \left(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla w_h,\ \boldsymbol{\beta}(u)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h \right)_E$. Then

$$\begin{aligned}
\tau_{33} &= \sum_{E\in\mathcal{T}_h} \delta_E \left(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla w_h,\ \left[\boldsymbol{\beta}(\Pi^0_p u_I)-\boldsymbol{\beta}(u)\right]\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h + \boldsymbol{\beta}(u)\cdot\left[\mathbf{\Pi}^0_{p-1}\nabla v_h - \nabla v_h\right] \right)_E \\
&\le C\,\mathcal{Q}(C\lambda) \sum_{E\in\mathcal{T}_h} \delta_E \,\|\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla w_h\|_E \left[\,\|\Pi^0_p u_I - u\|_{L^3(E)}\,\|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{L^3(E)}\right. \\
&\qquad\qquad \left. + \|\mathbf{\Pi}^0_{p-1}\nabla v_h - \nabla v_h\|_E\,\right] \\
&\le C\,\mathcal{Q}(C\lambda) \sum_{E\in\mathcal{T}_h} \delta_E\,\epsilon\, h_E^{-1}\, c_{inv}|w_h|_{1,E} \left[\,\|\Pi^0_p u_I - u\|_{1,E}\,\|\mathbf{\Pi}^0_{p-1}\nabla v_h\|_{1,E} + |\Pi^\nabla_k v_h - v_h|_{1,E}\,\right] \\
&\le C\,\mathcal{Q}(C\lambda) \sum_{E\in\mathcal{T}_h} \delta_E\,\epsilon\, h_E^{-1}\, c_{inv}|w_h|_{1,E} \left[\,\|\Pi^0_p u_I - u\|_{1,E}\, h_E^{-1}|v_h|_{1,E} + |v_h|_{1,E}\,\right] \\
&\le C\,\mathcal{Q}(C\lambda)\, h \left[\, c_{inv}\, h^{s-1}\,|u|_{s+1,\Omega} + h\,\right] |w_h|_{1,\Omega}\,|v_h|_{1,\Omega}.
\end{aligned} \tag{4.3.59}$$

Estimating τ_{34} in a similar way as τ_{32}, we obtain,

$$\tau_{34} \;\le\; C\,[\mathcal{Q}(C\lambda)]^2\, h \left[\, 2\,h^s\,|u|_{1+s,\Omega} + h + 1\,\right] |w_h|_{1,\Omega}\,|v_h|_{1,\Omega}. \tag{4.3.60}$$

Substituting (4.3.57), (4.3.58), (4.3.59) and (4.3.60) into the equation (4.3.56), we get,

$$\tau_3 \le C\,h\,\mathcal{C}_{43}\,|w_h|_{1,\Omega}\,|v_h|_{1,\Omega}, \tag{4.3.61}$$

where $\mathcal{C}_{43} := \mathcal{Q}(C\lambda)\left[\, h^{1-s}\,|u|_{s+1,\Omega}\,(1 + c_{inv} + \|\nabla\cdot\mathbf{\Pi}^0_{p-1}\nabla u\|_{\infty,E}) + h + (h^2+h)\|\nabla\cdot \mathbf{\Pi}^0_{p-1}\nabla u\|_{\infty,E}\,\right] + 2\,[\mathcal{Q}(C\lambda)]^2\left[\, 2h^s\,|u|_{s+1,\Omega} + h + 1\,\right]$.
Applying the results (4.3.50), (4.3.55) and (4.3.61), we get the required estimate (4.3.42) with $\mathcal{C}_4 := \mathcal{C}_{41} + \mathcal{C}_{42} + \mathcal{C}_{43}$. □

In our next lemma, we show that the operator M_h is in fact locally Lipschitz continuous. Let X be a Banach space over the domain ω and for any $\kappa > 0$, $y \in X$, we denote $B(y,\kappa) := \{\, z \in X \,:\, |z-y|_{1,\omega} \le \kappa\,\}$.

Lemma 4.7. *Consider* (4.1.2)*, Assumptions 4.2-4.4(section 1.3),* (4.2.12) *and* $(\mu, u_\mu) \in \mathcal{B}$ *with virtual interpolant* $u_{I,\mu}$*. For* $u_1,\ u_2 \in B(u_{I,\mu},\rho)\cap V_h^p$ *with* $\rho \le Ch^q$, $q > 1$ *and for sufficiently small* h*, the following estimate is obtained,*

$$\|M_h(\mu,u_1) - M_h(\mu,u_2) - \widehat{D}M_h(\mu,u_{I,\mu})(u_1-u_2)\|_* \le N_h(\rho, |u_{I,\mu}|_{1,\Omega})|u_1-u_2|_{1,\Omega},$$

where the function $N_h(0,\cdot)$ *satisfies,*

$$\lim_{h\to 0}\sup_{\mu\in I} N_h(0,\,\theta) = 0 \qquad \forall\theta\in\mathbb{R}^+. \tag{4.3.62}$$

Proof. For notational convenience, as usual we denote $u := u_\mu$ and $u_I := u_{I,\mu}$. First, we show the L^∞ bound for elements in $B(u_I,\rho)\cap V_h^k$.
Let $v_h \in B(u_I,\rho)\cap V_h^p$. Estimating similarly as in Remark 4.5, we get

$$\begin{aligned}
\sum_{E\in\mathcal{T}_h} \|\Pi_p^0 v_h\|_{L^\infty(E)} &\le C \sum_{E\in\mathcal{T}_h} \big(h_E^{-1}\|\Pi_p^0 v_h\|_E + |\Pi_p^0 v_h|_{1,E} + h_E|\Pi_p^0 v_h|_{2,E} \big) \\
&\le C \sum_{E\in\mathcal{T}_h} (2+c_{inv})\,(\,|v_h-u_I|_{1,E} + |u_I|_{1,E}\,) \\
&\le C\,(\,|v_h-u_I|_{1,\Omega} + |u_I|_{1,\Omega}\,) \le C\,(\,h^q + 2|u|_{1,\Omega}\,) \\
&\le C\,3\,\lambda \le C\,\lambda. \quad (\text{ for sufficiently small } h) \qquad (4.3.63)
\end{aligned}$$

Let us denote $\xi := u_1 - u_2$. Consider the inner product

$$\langle\, M_h(\mu,u_1) - M_h(\mu,u_2) - \widehat{D}M_h(\mu,u_I)\xi,\ v_h\,\rangle. \qquad (4.3.64)$$

We estimate each terms in the expansion of (4.3.64) one by one.
Let $\mathcal{J}_1 := \mu\big(\,b_h(u_1;u_1,v_h) - b_h(u_2;u_2,v_h) - \widehat{D}_1(\xi;u_I,v_h)\big)$. Adding and subtracting $\mu\sum_{E\in\mathcal{T}_h}\big(\,[\,\boldsymbol{\beta}(\Pi_p^0u_1)+\boldsymbol{\beta}(\Pi_p^0u_2)\,]\cdot\mathbf{\Pi}_{p-1}^0\nabla u_2,\ \delta_E\,\boldsymbol{\beta}(\Pi_p^0u_1)\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\,\big)_E$, we get,

$$\begin{aligned}
\mathcal{J}_1 &= \mu\sum_{E\in\mathcal{T}_h}\Big\{\ \big(\boldsymbol{\beta}(\Pi_p^0u_1)\cdot\mathbf{\Pi}_{p-1}^0\nabla(u_1-u_2),\ \delta_E\,\boldsymbol{\beta}(\Pi_p^0u_1)\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\,\big)_E \\
&\quad + \big(\,[\,\boldsymbol{\beta}(\Pi_p^0u_1)-\boldsymbol{\beta}(\Pi_p^0u_2)\,]\cdot\mathbf{\Pi}_{p-1}^0\nabla u_2,\ \delta_E\,\boldsymbol{\beta}(\Pi_p^0u_1)\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\,\big)_E \\
&\quad + \big(\,\boldsymbol{\beta}(\Pi_p^0u_2)\cdot\mathbf{\Pi}_{p-1}^0\nabla u_2,\ \delta_E\,[\,\boldsymbol{\beta}(\Pi_p^0u_1)-\boldsymbol{\beta}(\Pi_p^0u_2)\,]\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\,\big)_E \\
&\quad + \delta_E\,\tilde{\mathbb{S}}^2\,S^E((I-\Pi_p^\nabla)\xi,\,(I-\Pi_p^\nabla)v_h)\,\Big\} - \widehat{D}_1(\xi;u_I,v_h). \\
&\le \mu C\sum_{E\in\mathcal{T}_h}\Big[\Big\{\ \big(\boldsymbol{\beta}(\Pi_p^0u_1)\cdot\mathbf{\Pi}_{p-1}^0\nabla\xi,\ \delta_E\,\boldsymbol{\beta}(\Pi_p^0u_1)\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\,\big)_E \\
&\quad + \big(-\boldsymbol{\beta}(\Pi_p^0u_I)\cdot\mathbf{\Pi}_{p-1}^0\nabla\xi,\ \delta_E\,\boldsymbol{\beta}(\Pi_p^0u_I)\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\,\big)_E\ \Big\} \\
&\quad + \Big\{\ \big(\,\partial_u\boldsymbol{\beta}(\Pi_p^0x_1)\cdot\mathbf{\Pi}_{p-1}^0\nabla u_2,\ \delta_E\,\boldsymbol{\beta}(\Pi_p^0u_1)\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\,\big)_E \\
&\qquad - \big(\,\Pi_p^0\xi\,\partial_u\boldsymbol{\beta}(\Pi_p^0u_I)\cdot\mathbf{\Pi}_{p-1}^0\nabla u_I,\ \delta_E\boldsymbol{\beta}(\Pi_p^0u_I)\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\,\big)_E\ \Big\} \\
&\quad + \Big\{\ \big(\,\boldsymbol{\beta}(\Pi_p^0u_2)\cdot\mathbf{\Pi}_{p-1}^0\nabla u_2,\ \delta_E\,\partial_u\boldsymbol{\beta}(\Pi_p^0x_1)\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\,\big)_E \\
&\qquad - \big(\,\boldsymbol{\beta}(\Pi_p^0u_I)\cdot\mathbf{\Pi}_{p-1}^0\nabla u_I,\ \delta_E\Pi_p^0\xi\,\partial_u\boldsymbol{\beta}(\Pi_p^0u_I)\cdot\mathbf{\Pi}_{p-1}^0\nabla v_h\,\big)_E\ \Big\}\Big] \\
&:= R_1 + R_2 + R_3. \qquad (4.3.65)
\end{aligned}$$

where $x_1 := \gamma\,u_1 + (1-\gamma)u_2$ for some $\gamma\in(0,1)$.

Note that $x_1 \in B(u_I, \rho) \cap V_h^k$. Using (4.1.2), (4.2.5) and Hölder's inequality, we get,

$$R_1 \;\; \leq \;\; \mu\, C\, \delta\, |\xi|_{1,\Omega}\, |v_h|_{1,E}. \tag{4.3.66}$$

To R_2 adding and subtracting the term

$$\mu \sum_{E\in\mathcal{T}_h} \big(\Pi_p^0 \xi \,[\, \partial_u \boldsymbol{\beta}(\Pi_p^0 x_1) + \partial_u \boldsymbol{\beta}(\Pi_p^0 u_I)\,] \cdot \mathbf{\Pi}_{p-1}^0 \nabla u_I,\;\; \delta_E\, \boldsymbol{\beta}(\Pi_p^0 u_1) \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E$$

we obtain

$$\begin{aligned} R_2 \;\; = \;\; \mu\, C \sum_{E\in\mathcal{T}_h} \Big\{ & \big(\, \partial_u \boldsymbol{\beta}(\Pi_p^0 x_1) \cdot \mathbf{\Pi}_{p-1}^0 \nabla (u_2 - u_I),\;\; \delta_E\, \boldsymbol{\beta}(\Pi_p^0 u_1) \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E \\ & + \big(\, [\, \partial_u \boldsymbol{\beta}(\Pi_p^0 x_1) - \partial_u \boldsymbol{\beta}(\Pi_p^0 u_I)\,] \cdot \mathbf{\Pi}_{p-1}^0 \nabla u_I,\;\; \delta_E\, \boldsymbol{\beta}(\Pi_p^0 u_1) \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E \\ & + \big(\Pi_p^0 \xi\, \partial_u \boldsymbol{\beta}(\Pi_p^0 u_I) \cdot \mathbf{\Pi}_{p-1}^0 \nabla u_I,\;\; \delta_E [\, \boldsymbol{\beta}(\Pi_p^0 u_1) - \boldsymbol{\beta}(\Pi_p^0 u_I)\,] \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E \;\Big\} \end{aligned}$$

Using (4.1.2), generalised Hölder's inequality, (4.3.6), (2.6,[82]) and Poincaré inequality, we get,

$$\begin{aligned} R_2 \;\; \leq \;\; & \mu\, C \sum_{E\in\mathcal{T}_h} \delta_E \big(\|\mathbf{\Pi}_{p-1}^0 \nabla (u_2 - u_I)\|_{1,E} + \|x_1 - u_I\|_{1,E}\, \|\mathbf{\Pi}_{p-1}^0 \nabla u_I\|_{1,E} \\ & \qquad + \|u_1 - u_I\|_{1,E}\, \|\mathbf{\Pi}_{p-1}^0 \nabla u_I\|_{1,E} \big)\, \|\xi\|_{1,E}\, |v_h|_{1,E} \\ \leq \;\; & \mu\, C\, \delta \big(h^{-1}\, |u_2 - u_I|_{1,\Omega} + [\, |x_1 - u_I|_{1,\Omega} + |u_1 - u_I|_{1,\Omega}\,]\, h^{-1}\, |u_I|_{1,\Omega} \big)\, |\xi|_{1,\Omega}\, |v_h|_{1,\Omega} \end{aligned}$$

For any member $z_h \in B(u_I, \rho) \cap V_h^p$, we have,

$$|z_h|_{1,\Omega} \leq \rho + |u_I|_{1,\Omega} \text{ and } |u_I - z_h|_{1,\Omega} \leq \rho. \tag{4.3.67}$$

Thus, we obtain,

$$R_2 \;\; \leq \;\; \mu\, C\, \delta\, h^{-1}\, \rho\, (\, 1 + 2|u_I|_{1,\Omega}\,)\, |\xi|_{1,\Omega}\, |v_h|_{1,\Omega}. \tag{4.3.68}$$

To R_3 adding and subtracting the term

$$\mu \sum_{E\in\mathcal{T}_h} \big(\Pi_p^0 \xi\, \boldsymbol{\beta}(\Pi_p^0 u_2) \cdot \mathbf{\Pi}_{p-1}^0 \nabla u_I,\;\; \delta_E\, [\, \partial_u \boldsymbol{\beta}(\Pi_p^0 x_1) + \partial_u \boldsymbol{\beta}(\Pi_p^0 u_I)\,] \cdot \mathbf{\Pi}_{p-1}^0 \nabla v_h \big)_E$$

we obtain the same result as (4.3.68). Therefore substituting the results (4.3.66) and (4.3.68) into (4.3.65), we get,

$$\mathcal{J}_1 \leq \mu\, C\, \delta \left(1 + 2h^{-1} \rho \left(1 + 2|u_I|_{1,\Omega} \right) \right) |\xi|_{1,\Omega}\, |v_h|_{1,\Omega}. \tag{4.3.69}$$

Let $\mathcal{J}_2 := \mu\big(c_h(u_1; u_1, v_h) - c_h(u_2; u_2, v_h) - \widehat{D}_2(\xi; u_I, v_h) \big)$. Adding and subtracting $\mu \sum_{E\in\mathcal{T}_h} \big(\boldsymbol{\beta}(\Pi^0_p u_2) \cdot \mathbf{\Pi}^0_{p-1} \nabla u_1,\ \Pi^0_p v_h \big)_E$ to $\mathcal{J}_2$, we get,

$$\begin{aligned}
\mathcal{J}_2 \;&=\; \mu \sum_{E\in\mathcal{T}_h} \Big\{ \big(r(\Pi^0_p u_1) - r(\Pi^0_p u_2),\ \Pi^0_p v_h \big)_E + \big([\boldsymbol{\beta}(\Pi^0_p u_1) - \boldsymbol{\beta}(\Pi^0_p u_2)] \cdot \mathbf{\Pi}^0_{p-1} \nabla u_1,\ \Pi^0_p v_h \big)_E \\
&\qquad + \big(\boldsymbol{\beta}(\Pi^0_p u_2) \cdot \mathbf{\Pi}^0_{p-1} \nabla \xi,\ \Pi^0_p v_h \big)_E - \widehat{D}_2(\xi; u_I, v_h) \Big\} \\
&\leq\; \mu \sum_{E\in\mathcal{T}_h} \Big\{ \big(r(\Pi^0_p u_1) - r(\Pi^0_p u_2),\ \Pi^0_p v_h \big)_E + \big(\Pi^0_p \xi\, \partial_u \boldsymbol{\beta}(\Pi^0_p w_1) \cdot \mathbf{\Pi}^0_{p-1} \nabla u_1,\ \Pi^0_p v_h \big)_E \\
&\qquad + \big(\boldsymbol{\beta}(\Pi^0_p u_2) \cdot \mathbf{\Pi}^0_{p-1} \nabla \xi,\ \Pi^0_p v_h \big)_E - \widehat{D}_2(\xi; u_I, v_h) \Big\},
\end{aligned}$$

where $\quad w_1 := \nu\, u_1 + (1-\nu) u_2$ for some $\nu \in (0,1)$.
Again, adding & subtracting $\mu \sum_{E\in\mathcal{T}_h} \big(\Pi^0_p \xi\, \boldsymbol{\beta}(\Pi^0_p u_I) \cdot \mathbf{\Pi}^0_{p-1} \nabla u_1,\ \Pi^0_p v_h \big)_E$ we get,

$$\begin{aligned}
\mathcal{J}_2 \;\leq\; \mu \sum_{E\in\mathcal{T}_h} \Big\{ & \big(\Pi^0_p \xi\, [\partial_u r(\Pi^0_p w_2) - \partial_u r(\Pi^0_p u_I)],\ \Pi^0_p v_h \big)_E \\
& + \big(\Pi^0_p \xi\, [\partial_u \boldsymbol{\beta}(\Pi^0_p w_1) - \partial_u \boldsymbol{\beta}(\Pi^0_p u_I)] \cdot \mathbf{\Pi}^0_{p-1} \nabla u_1,\ \Pi^0_p v_h \big)_E \\
& + \big(\Pi^0_p \xi\, \partial_u \boldsymbol{\beta}(\Pi^0_p u_I) \cdot \mathbf{\Pi}^0_{p-1} \nabla (u_I - u_1),\ \Pi^0_p v_h \big)_E \\
& + \big(\Pi^0_p (u_2 - u_I)\, \partial_u \boldsymbol{\beta}(\Pi^0_p w_3) \cdot \mathbf{\Pi}^0_{p-1} \nabla \xi,\ \Pi^0_p v_h \big)_E \Big\}
\end{aligned}$$

where, $\quad w_2 := \bar{\nu}\, u_2 + (1-\bar{\nu}) u_I,\ w_3 := \tilde{\nu}\, u_1 + (1-\tilde{\nu}) u_2$ for some $\bar{\nu},\ \tilde{\nu} \in (0,1)$.

Note that $w_1,\ w_2,\ w_3 \in B(u_I, \rho) \cap V^p_h$. Using (4.1.2), generalised Hölder's inequality, (4.3.6) and Poincaré inequality, we get,

$$\begin{aligned}
\mathcal{J}_2 \;&\leq\; \mu\, C \sum_{E\in\mathcal{T}_h} \big\{ \|w_2 - u_I\|_E + \|w_1 - u_I\|_E\, |u_1|_{1,E} + |u_I - u_1|_{1,E} + \|u_2 - u_I\|_E \big\}\, \|\xi\|_{1,E}\, \|v_h\|_{1,E} \\
&\leq\; \mu\, C \big\{ |w_2 - u_I|_{1,\Omega} + |w_1 - u_I|_{1,\Omega}\, |u_1|_{1,\Omega} + |u_I - u_1|_{1,\Omega} + |u_2 - u_I|_{1,\Omega} \big\}\, |\xi|_{1,\Omega}\, |v_h|_{1,\Omega}
\end{aligned}$$

Therefore, using (4.3.67),

$$\mathcal{J}_2 \leq \mu\, C\, \rho \left(3 + \rho + |u_I|_{1,\Omega} \right) |\xi|_{1,\Omega}\, |v_h|_{1,\Omega} \tag{4.3.70}$$

Let $\mathcal{J}_3 := \mu\big(d_h(u_1; u_1, v_h) - d_h(u_2; u_2, v_h) - \widehat{D}_3(\xi; u_I, v_h) \big)$. Adding and subtracting

$\mu \sum_{E\in\mathcal{T}_h} \big(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla u_2 + r(\Pi^0_p u_2),\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_1)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E$ to $\mathcal{J}_3$, we get,

$$
\begin{aligned}
\mathcal{J}_3 \;&=\; \mu \sum_{E\in\mathcal{T}_h} \Big\{ \big(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla\xi + [r(\Pi^0_p u_1) - r(\Pi^0_p u_2)],\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_1)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E \\
&\qquad + \big(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla u_2 + r(\Pi^0_p u_2),\ \delta_E\,[\boldsymbol{\beta}(\Pi^0_p u_1) - \boldsymbol{\beta}(\Pi^0_p u_2)]\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E \\
&\qquad\qquad - \widehat{D}_3(\xi; u_I, v_h)\Big\} \\
&\le\; \mu\, C \sum_{E\in\mathcal{T}_h} \Big[\Big\{ \big(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla\xi + \Pi^0_p\xi\,\partial_u r(\Pi^0_p y_1),\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_1)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E \\
&\qquad + \big(r(\Pi^0_p u_2),\ \delta_E\,\Pi^0_p\xi\,\partial_u\boldsymbol{\beta}(\Pi^0_p y_2)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E \\
&\qquad + \big(r(\Pi^0_p u_I),\ \delta_E\,\Pi^0_p\xi\,\partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E \\
&\qquad + \big(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla\xi + \Pi^0_p\xi\,\partial_u r(\Pi^0_p u_I),\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E \Big\} \\
&\qquad + \Big\{ \big(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla u_2,\ \delta_E\,\partial_u\boldsymbol{\beta}(\Pi^0_p y_2)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E \\
&\qquad + \big(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla u_I,\ \delta_E\,\Pi^0_p\xi\,\partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E \Big\} \Big] \\
&:=\; Y_1 + Y_2. \qquad (4.3.71)
\end{aligned}
$$

where $y_1 := \bar{\gamma}\,u_1 + (1-\bar{\gamma})u_2,\ y_2 := \tilde{\gamma}\,u_1 + (1-\tilde{\gamma})\,u_2$ for some $\bar{\gamma},\ \tilde{\gamma} \in (0,1)$.
Using (4.1.2), Cauchy-Schwarz ineq., (4.3.1), (4.2.12) and Hölder's ineq. in Y_1, we get,

$$
\begin{aligned}
Y_1 \;&\le\; \mu\, C \sum_{E\in\mathcal{T}_h} \big(2\,h_E\,c_{inv}\,|\xi|_{1,E} + 3\,\delta_E\,\|\xi\|_E \big)\,|v_h|_{1,E} \\
&\le\; \mu\, C\,\big(2\,c_{inv}\,h + 3\,\delta \big)\,|\xi|_{1,\Omega}\,|v_h|_{1,\Omega}. \quad (\text{ use Poincaré inequality }) \quad (4.3.72)
\end{aligned}
$$

Adding & subtracting $\mu \sum_{E\in\mathcal{T}_h} \big(-\nabla\cdot\epsilon\,\mathbf{\Pi}^0_{p-1}\nabla u_I,\ \delta_E\,\Pi^0_p\xi\,\partial_u\boldsymbol{\beta}(\Pi^0_p y_2)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E$ to Y_2 we get

$$
\begin{aligned}
Y_2 \;&=\; \mu\, C \sum_{E\in\mathcal{T}_h} \Big\{ \big(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla(u_2 - u_I),\ \delta_E\,\Pi^0_p\xi\,\partial_u\boldsymbol{\beta}(\Pi^0_p y_2)\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E \\
&\qquad + \big(-\nabla\cdot\epsilon\mathbf{\Pi}^0_{p-1}\nabla u_I,\ \delta_E\,\Pi^0_p\xi\,[\,\partial_u\boldsymbol{\beta}(\Pi^0_p y_2) - \partial_u\boldsymbol{\beta}(\Pi^0_p u_I)\,]\cdot\mathbf{\Pi}^0_{p-1}\nabla v_h\big)_E \Big\}
\end{aligned}
$$

Using (4.1.2), generalised Hölder's inequality, (4.3.1), (4.3.6) and (2.6[82]), we get,

$$
\begin{aligned}
Y_2 \;&\le\; \mu\, C \sum_{E\in\mathcal{T}_h} \delta_E\,\epsilon\,c_{inv}\,h_E^{-1}\,\big(|u_2 - u_I|_{1,E} + \|y_2 - u_I\|_{1,E}\big)\,\|\xi\|_{1,E}\,\|\mathbf{\Pi}^0_k\nabla v_h\|_{1,E} \\
&\le\; \mu\, C \sum_{E\in\mathcal{T}_h} \delta_E\,\epsilon\,c_{inv}\,h_E^{-2}\,\big(|u_2 - u_I|_{1,E} + \|y_2 - u_I\|_{1,E}\big)\,\|\xi\|_{1,E}\,|\nabla v_h|_{1,E} \\
&\le\; \mu\, C\,c_{inv}\,\big(|u_2 - u_I|_{1,\Omega} + |y_2 - u_I|_{1,\Omega}\big)\,|\xi|_{1,\Omega}\,|v_h|_{1,\Omega} \quad (\text{ use (A5,i) }) \\
&\le\; \mu\, C\,c_{inv}\,2\,\rho\,|\xi|_{1,\Omega}\,|v_h|_{1,\Omega}. \quad (\text{ use (4.3.67) }) \qquad (4.3.73)
\end{aligned}
$$

Substituting (4.3.72) and (4.3.73) into (4.3.71), we get,

$$\mathcal{J}_3 \leq \mu\, C\, (\, 2\, c_{inv}\, h + 3\,\delta + c_{inv}\, 2\,\rho\,)\, |\xi|_{1,\Omega}\, |v_h|_{1,\Omega}. \tag{4.3.74}$$

Using the results (4.3.69), (4.3.70) and (4.3.74) we obtain

$$\begin{aligned} N_h(\rho, |u_I|_{1,\Omega}) \;\; &:= \;\; \mu\, C\, \big[\, 4\,\delta + 2c_{inv}\, h + \delta\, h^{-1}\, \rho\, (\, 1 + 2|u_I|_{1,\Omega}\,) \\ &\qquad + \rho\, (\, 3 + \rho + |u_I|_{1,\Omega} + 2c_{inv}\,)\, \big]. \end{aligned} \tag{4.3.75}$$

The results (4.3.69), (4.3.70) and (4.3.74), along with using (4.2.12), putting $\rho = 0$ in (4.3.75), we validate the assertions of the lemma. □

4.4 Convergence analysis

Let u_h be the discrete solution to (4.2.13). We use the following natural norm for our analysis,

$$|\!|\!| v_h |\!|\!|^2 = \epsilon\, |v_h|^2_{1,\Omega} \, + \sum_{E \in \mathcal{T}_h} \delta_E\, \|\boldsymbol{\beta}(u_h) \cdot \nabla v_h\|^2_E. \tag{4.4.1}$$

We have proved the existence and uniqueness of discrete solution using the results given in Section 3.4 of [80].

Theorem 4.1. *Consider the following estimations*

(i) $\quad \|T_h\|_{L(Y_h, V_h^k)} \leq C, \quad$ *and* $\quad \lim_{h \to 0} \|T - T_h\|_{L(Y_h, V_h^k)} = 0. \quad$ (4.4.2)

(ii) $\quad \lim_{h \to 0} |v - v_I|_{1,\Omega} = 0 \quad \forall v \in H^1_0(\Omega), \quad$ *and* $\quad$ (4.4.3)

$$\lim_{h \to 0} \sup_{\mu \in I} \|M_h(\mu, u_I(\mu)) - M_0(\mu, u(\mu))\|_* = 0. \tag{4.4.4}$$

(iii) $\quad \forall\, w_h \in V_h^k \quad \exists\, \widehat{D}M_h(\mu, w_h) \in L(V_h^k, Y_h),$

$$\lim_{h \to 0} \sup_{\mu \in I} \|\widehat{D}M_h(\mu, u_{I,\mu}) - DM_0(\mu, u(\mu))\|_{L(V_h^k, Y_h)} = 0. \tag{4.4.5}$$

(iv) $\quad$ *for any* $z_h,\, w_h \in B(u_{I,\mu}, \rho) \cap V_h^k$ *we have*

$$\begin{aligned} &\|M_h(\mu, z_h) - M_h(\mu, w_h) - \widehat{D}M_h(u_{I,\mu})(z_h - w_h)\|_{L(V_h^k, Y_h)} \\ &\qquad \leq N_h(\rho, |u_I|_{1,\Omega}) |z_h - w_h|_{1,\Omega}, \end{aligned} \tag{4.4.6}$$

where $N_h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and monotonically increasing in each variable and satisfy* $\lim_{h \to 0} N_h(\, 0,\, \theta) = 0 \quad \forall\, \theta \in \mathbb{R}^+.$ $\quad$ (4.4.7)

Assume that the conditions (4.4.2)-(4.4.7) *are satisfied. Then,*

1. there exists a neighbourhood $\mathcal{N}_0$ *around the origin in* $H_0^1(\Omega)$ *and for sufficiently small* h*, a unique branch* $\{\,(\,\mu,\, u_{h\mu}\,) :\ \mu \in I,\ u_{h,\mu} \in V_h^p\,\}$ *of nonsingular solutions of* (4.2.19)*, such that*

$$\forall\, \mu \in I, \qquad u_\mu - u_{h,\mu} \in \mathcal{N}_0 \qquad \textit{and} \tag{4.4.8}$$

2. the following error estimate holds,

$$\begin{aligned}\|u_\mu - u_{h,\mu}\|_{1,\Omega} \leq \ \ C \quad & \big\{\, \|u - u_I\|_{1,\Omega} + \|\mu\,(T - T_h) M_0(\mu, u_\mu)\|_{1,\Omega} \\ & \quad + \|M_h(\mu, u_{I,\mu}) - M_0(\mu, u_\mu)\|_* \,\big\}. \end{aligned} \tag{4.4.9}$$

Proof. Consider the discrete operator T_h defined in (4.2.18). For any $g \in Y_h$, we have $T_h g \in V_h^k$ and from (4.2.18) we note

$$\min\{1, \alpha_*\}\, |T_h g|_{1,\Omega}^2 \leq \langle\, g,\, T_h g\,\rangle \leq \|g\|_\Omega\, |T_h g|_{1,\Omega}.$$

Thus $|T_h g|_{1,\Omega} \leq \|g\|_\Omega \ \ \forall\, g \in Y_h$ implies that $\|T_h\|_{Y_h, V_h^k)} \leq 1$.

Considering the continuity and coercivity property of (4.1.5), definition (4.2.18) and applying Cea's lemma we get (similar to (4.19) in [81]),

$$\lim_{h \to 0} |v - v_I|_{1,\Omega} = 0 \qquad \forall v \in H_0^1(\Omega).$$

Note that the results (4.4.3) and (4.4.4) follows from (4.3.4) and Lemma 4.5, respectively. Similarly the estimates (4.4.5) and (4.4.6) are obtained as a consequence of Lemma 4.6 and Lemma 4.7, respectively.

Thus, the existence of a unique branch $\{\,(\,\mu,\, u_h(\mu)\,) :\ \mu \in I,\ u_h(\mu) \in V_h\,\}$ of nonsingular solutions of (4.2.19) satisfying (4.4.8) is guaranteed.

For sufficiently smooth Tf, from the definitions of $T,\ T_h,$ Cea's lemma and by using (4.3.4), we obtain an estimate

$$|(T_h - T) f|_{1,\Omega} \leq C\, h^k\, |Tf|_{k+1,\Omega}. \tag{4.4.10}$$

In (4.4.9), applying (4.4.3), boundedness of $T - T_h$ in (4.4.2) and lemma 4.5, we get

$$\|u_\mu - u_h(\mu)\|_{1,\Omega} \leq C\, h^s\, |u|_{s+1,\Omega}. \tag{4.4.11}$$

□

Next, we prove two auxiliary lemmas that will be used in convergence estimates with respect to $|||\cdot|||$.

Lemma 4.8. *Let* $u_h \in V_h^p$ *be the discrete solution of* (4.2.13) *and* u_I *be the virtual element interpolant of an exact solution* u. *Then using* (4.1.2)*, assumptions 4.2-4.4,* (4.2.12) *and for* $\phi_h = u_I - u_h$ *the following estimate is obtained,*

$$\mathcal{A}(\{u_I, u_h\}\,; u_I, \phi_h) - \mathcal{A}(\{u_h, u_h\}\,; u_h, \phi_h) \geq \mathcal{C}_{51}\,|||\phi_h|||^2 - \mathcal{C}_{52}\,|\phi_h|^2_{1,\Omega}. \tag{4.4.12}$$

where $\mathcal{C}_{51} := \min\{1, \alpha_*, \wp_*\}$ *and* $\mathcal{C}_{52} := C\,\mathcal{Q}(\lambda)\big[\,\lambda\mathcal{Q}(\lambda)\delta + 1 + \lambda + c_{inv}h + \delta\,\big]$.

Proof. First we consider $\mathcal{A}_1 := a_h(u_I, \phi_h) - a_h(u_h, \phi_h) = a_h(\phi_h, \phi_h)$. Then using (4.2.5) and the inequality $\|(I - \mathbf{\Pi}^0_{p-1})\nabla\phi_h\|_E \leq \|\nabla(I - \Pi^\nabla_p)\phi_h\|_E$ (see [12]), we get,

$$\mathcal{A}_1 \geq \min\{1,\ \alpha_*\}\,\epsilon\,|\phi_h|^2_{1,\Omega}. \tag{4.4.13}$$

Along the lines of remark 4.5, we get the estimate,

$$\|\Pi^0_p u_h\|_{\infty,E} \leq C\,\lambda. \tag{4.4.14}$$

Second, let $\mathcal{A}_2 := b_h(\{u_I, u_h\}; u_I, \phi_h) - b_h(\{u_h, u_h\}; u_h, \phi_h)$. Adding and subtracting the term $\sum_{E\in\mathcal{T}_h} \big(\boldsymbol{\beta}(\Pi^0_p u_h)\cdot\mathbf{\Pi}^0_{p-1}\nabla u_I,\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_h)\cdot\mathbf{\Pi}^0_{p-1}\nabla\phi_h\big)_E$, we get,

$$\begin{aligned}
\mathcal{A}_2 &:= b_h(u_h;\phi_h,\phi_h) + \sum_{E\in\mathcal{T}_h}\big([\boldsymbol{\beta}(\Pi^0_p u_I) - \boldsymbol{\beta}(\Pi^0_p u_h)]\cdot\mathbf{\Pi}^0_{p-1}\nabla u_I,\ \delta_E\,\boldsymbol{\beta}(\Pi^0_p u_h)\cdot\mathbf{\Pi}^0_{p-1}\nabla\phi_h\big)_E \\
&\geq \wp_*\,b(\{u_h,u_h\};\phi_h,\phi_h) + A_{21}. \qquad (\text{ use (4.2.10) })
\end{aligned} \tag{4.4.15}$$

Using the procedure in remark 4.5 we get the estimate,

$$\forall E \in \mathcal{T}_h, \qquad \|\Pi^0_p\phi_h\|_{\infty,E} \leq C\,|\phi_h|_{1,E}. \tag{4.4.16}$$

Using (4.1.2) and the generalised Hölder's inequality (with $\frac{1}{\infty} = 0$), we get,

$$\begin{aligned}
|A_{21}| &\leq C\,[\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h}\delta_E\,\|\Pi^0_p\phi_h\|_{\infty,E}\,|u_I|_{1,E}\,|\phi_h|_{1,E} \\
&\leq C\,[\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h}\delta_E\,2\,|u|_{1,E}\,|\phi_h|^2_{1,E} \qquad (\text{ use (4.4.16) }) \\
&\leq C\,\lambda\,[\mathcal{Q}(\lambda)]^2\,\delta\,|\phi_h|^2_{1,\Omega}.
\end{aligned} \tag{4.4.17}$$

Thus, from (4.4.15) and (4.4.17), we get,

$$\mathcal{A}_2 \geq \wp_* \sum_{E\in\mathcal{T}_h} \delta_E \,\|\boldsymbol{\beta}(u_h)\cdot\nabla\phi_h\|_E^2 - C\,\lambda\,[\mathcal{Q}(\lambda)]^2\,\delta\,|\phi_h|_{1,\Omega}^2. \tag{4.4.18}$$

Third, we consider $\mathcal{A}_3 := c_h(u_I; u_I, \phi_h) - c_h(u_h; u_h, \phi_h)$. To this adding and subtracting the term $\sum_{E\in\mathcal{T}_h} \big(\boldsymbol{\beta}(\Pi_p^0 u_h)\cdot\mathbf{\Pi}_{p-1}^0\nabla u_I,\, \Pi_p^0\phi_h\big)_E$, we get,

$$\begin{aligned}
|\mathcal{A}_3| &= \Big|\sum_{E\in\mathcal{T}_h}\Big\{\ \big([r(\Pi_p^0 u_I) - r(\Pi_p^0 u_h)] + \boldsymbol{\beta}(\Pi_p^0 u_h)\cdot\mathbf{\Pi}_{p-1}^0\nabla\phi_h,\, \Pi_p^0\phi_h\big)_E \\
&\qquad + \big([\boldsymbol{\beta}(\Pi_p^0 u_I) - \boldsymbol{\beta}(\Pi_p^0 u_h)]\cdot\mathbf{\Pi}_{p-1}^0\nabla u_I,\, \Pi_p^0\phi_h\big)_E\ \Big\} \\
&\leq C\,\mathcal{Q}(\lambda)\sum_{E\in\mathcal{T}_h}\Big\{\|\phi_h\|_E^2 + \|\phi_h\|_E\,|\phi_h|_{1,E} + \|\Pi_p^0\phi_h\|_{L^\infty(E)}\,|u_I|_{1,E}\,\|\phi_h\|_E\Big\} \\
&\leq C\,\mathcal{Q}(\lambda)\sum_{E\in\mathcal{T}_h}\Big\{\|\phi_h\|_E^2 + \|\phi_h\|_E\,|\phi_h|_{1,E} + |\phi_h|_{1,E}\,|u_I|_{1,E}\,\|\phi_h\|_E\Big\} \\
&\leq C\,\mathcal{Q}(\lambda)\Big\{\|\phi_h\|_\Omega^2 + \|\phi_h\|_\Omega\,|\phi_h|_{1,\Omega} + |\phi_h|_{1,\Omega}\,2\,|u|_{1,\Omega}\,\|\phi_h\|_\Omega\Big\} \leq C\,\mathcal{Q}(\lambda)\,(1+\lambda)\,|\phi_h|_{1,\Omega}^2.
\end{aligned}$$

Thus we obtain the inequality,

$$\mathcal{A}_3 \geq -C\,\mathcal{Q}(\lambda)\,(1+\lambda)\,|\phi_h|_{1,\Omega}^2. \tag{4.4.19}$$

Lastly, we estimate $|\mathcal{A}_4| := |d_h(u_h; u_I, \phi_h) - d_h(u_h; u_h, \phi_h)|$. Then,

$$\begin{aligned}
|\mathcal{A}_4| &= \Big|\sum_{E\in\mathcal{T}_h}\Big\{\ \big(-\nabla\cdot\epsilon\,\mathbf{\Pi}_{p-1}^0\nabla\phi_h,\ \delta_E\,\boldsymbol{\beta}(\Pi_p^0 u_h)\cdot\mathbf{\Pi}_{p-1}^0\nabla\phi_h\big)_E \\
&\qquad + \big([r(\Pi_p^0 u_I) - r(\Pi_p^0 u_h)],\ \delta_E\,\boldsymbol{\beta}(\Pi_p^0 u_h)\cdot\mathbf{\Pi}_{p-1}^0\nabla\phi_h\big)_E\ \Big\}\Big| \\
&\leq C\,\mathcal{Q}(\lambda)\sum_{E\in\mathcal{T}_h}\big(c_{inv}\,h_E\,|\phi_h|_{1,E}^2 + \delta_E\,\|\phi_h\|_E\,|\phi_h|_{1,E}\big) \qquad (\text{ use } (4.3.1), (A1)\ ) \\
&\leq C\,\mathcal{Q}(\lambda)(\,c_{inv}\,h + \delta\,)\,|\phi_h|_{1,\Omega}^2.
\end{aligned}$$

Thus, $\qquad \mathcal{A}_4 \geq -C\,\mathcal{Q}(\lambda)(\,c_{inv}\,h + \delta\,)\,|\phi_h|_{1,\Omega}^2. \qquad$ (4.4.20)

Combining the results (4.4.13), (4.4.18), (4.4.19) and (4.4.20) we get the assertion (4.4.12). □

Lemma 4.9. *Let* $u_h \in V_h^p$ *be the discrete solution of* (4.2.13) *and* u_I *be the virtual element interpolant of an exact solution* u*. Then using* (4.1.2)*, Assumptions 4.2-4.4,* (4.2.12) *and* *for* $\phi_h = u_I - u_h$ *the following estimate is obtained,*

$$\mathcal{A}(\{u_I, u_h\}\,; u_I, \phi_h) - \mathcal{A}(\{u, u_h\}\,; u, \phi_h) \;\leq\; \mathcal{C}_6\, h^{2s}\,(\epsilon + h), \tag{4.4.21}$$

for some constant $\mathcal{C}_6 > 0$.

Proof. Let $\psi := u_I - u$. We have by (4.4.11) and (4.3.4),

$$|\phi_h|_{1,\Omega} \leq |\psi_h|_{1,\Omega} + |u - u_h|_{1,\Omega} \leq C\, h^s\, |u|_{s+1,\Omega} + C\, h^s \leq C\, h^s. \tag{4.4.22}$$

Similarly, using (4.3.4), we have,

$$|\psi_h|_{1,\Omega} \leq C\, h^s. \tag{4.4.23}$$

Consider, $\mathcal{Z}_1 := a_h(u_I, \phi_h) - a_h(u, \phi_h) = a_h(\psi_h, \phi_h)$. Then,

$$\begin{aligned} \mathcal{Z}_1 \;&\leq\; C \sum_{E\in\mathcal{T}_h} \big(\epsilon\, |\psi_h|_{1,E}\, |\phi_h|_{1,E} + \epsilon\, \alpha^*\, |\psi_h|_{1,E}\, |\phi_h|_{1,E} \big) \\ &\leq\; C\, \epsilon\, (1 + \alpha^*)\, |\psi_h|_{1,\Omega}\, |\phi_h|_{1,\Omega} \leq C\, \epsilon\, h^{2s} \quad \text{(use (4.4.22) and (4.4.23)).} \end{aligned} \tag{4.4.24}$$

Now, let $\mathcal{Z}_2 := b_h(\{u_I, u_h\}\,; u_I, \phi_h) - b_h(\{u, u_h\}\,; u, \phi_h)$. Adding and subtracting the term $\sum_{E\in\mathcal{T}_h} \big(\boldsymbol{\beta}(\Pi^0_p u) \cdot \boldsymbol{\Pi}^0_{p-1} \nabla u_I,\, \delta_E\, \boldsymbol{\beta}(\Pi^0_p u_h) \cdot \boldsymbol{\Pi}^0_{p-1} \nabla \phi_h \big)_E$, we get,

$$\begin{aligned} \mathcal{Z}_2 \;&=\; \sum_{E\in\mathcal{T}_h} \Big\{ \big([\boldsymbol{\beta}(\Pi^0_p u_I) - \boldsymbol{\beta}(\Pi^0_p u)] \cdot \boldsymbol{\Pi}^0_{p-1} \nabla u_I,\, \delta_E\, \boldsymbol{\beta}(\Pi^0_p u_h) \cdot \boldsymbol{\Pi}^0_{p-1} \nabla \phi_h \big)_E \\ &\qquad + \big(\boldsymbol{\beta}(\Pi^0_p u) \cdot \boldsymbol{\Pi}^0_{p-1} \nabla \psi_h,\, \delta_E\, \boldsymbol{\beta}(\Pi^0_p u_h) \cdot \boldsymbol{\Pi}^0_{p-1} \nabla \phi_h \big)_E + \delta_E\, \tilde{\mathcal{S}}^2\, S^E((I - \Pi^\nabla_p)\psi_h, (I - \Pi^\nabla_p)\phi_h) \Big\} \\ &\leq\; C\, [\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h} \delta_E \big(\|\psi_h\|_E\, |u_I|_{1,E}\, |\phi_h|_{L^\infty(E)} + (1 + \alpha^*)\, |\psi_h|_{1,E}\, |\phi_h|_{1,E} \big) \\ &\leq\; C\, [\mathcal{Q}(\lambda)]^2 \sum_{E\in\mathcal{T}_h} \delta_E\, (\, \lambda + 1 + \alpha^*\,)\, \|\psi_h\|_{1,E}\, |\phi_h|_{1,E} \quad \text{(use (4.4.16))} \\ &\leq\; C\, (\, \lambda + 1 + \alpha^*\,)\, \delta\, |\psi_h|_{1,\Omega}\, |\phi_h|_{1,\Omega} \quad \text{(use Hölder's and Poincaré inequality)} \\ &\leq\; C\, h\, h^{2s} \quad \text{(use (4.4.23), (4.4.22) and (4.2.12)).} \end{aligned} \tag{4.4.25}$$

Next, we consider $\mathcal{Z}_3 := c_h(u_I; u_I, \phi_h) - c_h(u; u, \phi_h)$. Adding and subtracting the term $\sum_{E\in\mathcal{T}_h} \big(\boldsymbol{\beta}(\Pi^0_p u) \cdot \boldsymbol{\Pi}^0_{p-1} \nabla u_I,\, \Pi^0_p \phi_h \big)_E$, we get,

$$\begin{aligned} \mathcal{Z}_3 \;&=\; \sum_{E\in\mathcal{T}_h} \Big\{ \big([r(\Pi^0_p u_I) - r(\Pi^0_p u)] + \boldsymbol{\beta}(\Pi^0_p u) \cdot \boldsymbol{\Pi}^0_{p-1} \nabla \psi_h,\, \Pi^0_p \phi_h \big)_E \\ &\qquad + \big([\boldsymbol{\beta}(\Pi^0_p u_I) - \boldsymbol{\beta}(\Pi^0_p u)] \cdot \boldsymbol{\Pi}^0_{p-1} \nabla u_I,\, \Pi^0_p \phi_h \big)_E \Big\}. \end{aligned}$$

Estimating the term $\mathcal{Z}_3$, we get,

$$\begin{aligned}
\mathcal{Z}_3 &\leq C\,\mathcal{Q}(\lambda)\,\mu_2 \sum_{E\in\mathcal{T}_h} \epsilon\left\{\,\|\psi_h\|_E\,\|\phi_h\|_E + \|\psi_h\|_E\,|u_I|_{1,E}\,\|\Pi_p^0\phi_h\|_{L^\infty(E)} + |\psi_h|_{1,E}\,\|\phi_h\|_E\,\right\} \\
&\leq C\,\mathcal{Q}(\lambda)\,\mu_2 \sum_{E\in\mathcal{T}_h} \epsilon\left\{\,\|\psi_h\|_E\,\|\phi_h\|_E + \|\psi_h\|_E\,|u_I|_{1,E}\,|\phi_h|_{1,E} + |\psi_h|_{1,E}\,\|\phi_h\|_E\,\right\} \\
&\leq C\,\epsilon\,(\,2+|u_I|_{1,\Omega}\,)\,|\psi_h|_{1,\Omega}\,|\phi_h|_{1,\Omega} \quad (\text{ use Hölder's and Poincaré inequality }) \\
&\leq C\,\epsilon\,h^{2s}. \qquad (4.4.26)
\end{aligned}$$

Finally, we have $\mathcal{Z}_4 := d_h(u_h; u_I, \phi_h) - d_h(u_h; u, \phi_h)$. Therefore,

$$\begin{aligned}
\mathcal{Z}_4 &= \sum_{E\in\mathcal{T}_h} \left(-\nabla\cdot\epsilon\boldsymbol{\Pi}_{p-1}^0\nabla\psi_h + [r(\Pi_p^0 u_I) - r(\Pi_p^0 u)],\ \delta_E\,\boldsymbol{\beta}(\Pi_p^0 u_h)\cdot\boldsymbol{\Pi}_{p-1}^0\nabla\phi_h\,\right)_E \\
&\leq C\,\mathcal{Q}(\lambda) \sum_{E\in\mathcal{T}_h} \delta_E\left(\,\epsilon\,c_{inv}\,h_E^2\,|\psi_h|_{1,E} + \|\psi_h\|_E\,\right)|\phi_h|_{1,E} \\
&\leq C\,h\,|\psi_h|_{1,\Omega}\,|\phi_h|_{1,\Omega} \leq C\,h\,h^{2s}. \qquad (4.4.27)
\end{aligned}$$

Combining the estimates (4.4.24)- (4.4.27), we get the desired result (4.4.21). □

Now using auxillary lemma 4.8 and lemma 4.9, we obtain the following error estimate theorem.

Theorem 4.2. *Let us consider* (4.1.2)*, Assumptions 4.2-4.4,* (4.2.12)*. Let* $u_h \in V_h^p$ *be the discrete solution to* (4.2.13) *and* $u \in H_0^1(\Omega)$ *be the exact solution satisfying* (4.1.3) *with* $u \in H^{s+1}(\Omega)$*. Then, for sufficiently small h, we have,*

$$|||u-u_h||| \leq C\,h^k\,(\sqrt{h}+\sqrt{\epsilon}). \qquad (4.4.28)$$

Proof. From lemma 4.8, lemma 4.9 and (4.4.22), we have

$$|||u_I-u_h|||^2 \leq \left(\frac{\mathcal{C}_{52}}{\epsilon_0\mathcal{C}_{51}} + \frac{\mathcal{C}_6}{\mathcal{C}_{51}}\right) h^{2s}\,(\epsilon+h). \qquad (4.4.29)$$

Next, using (4.3.4), (4.2.12) we get,

$$\begin{aligned}
|||u-u_I|||^2 &= \epsilon|u-u_I|_{1,\Omega}^2 + \sum_{E\in\mathcal{T}_h} \delta_E\,\|\boldsymbol{\beta}(u_h)\cdot\nabla(u-u_I)\|_E^2 \\
&\leq C\,\epsilon\,h^{2s}\,|u|_{1+s,\Omega}^2 + C\,[\mathcal{Q}(\lambda)]^2\,h^{2s+1}\,|u|_{1+s,\Omega}^2 \\
&\leq \mathcal{C}_7\,h^{2s}(\epsilon+h), \qquad (4.4.30)
\end{aligned}$$

where $\mathcal{C}_7 := C\,(1+[\mathcal{Q}(\lambda)]^2)\,|u|_{1+s,\Omega}^2$.

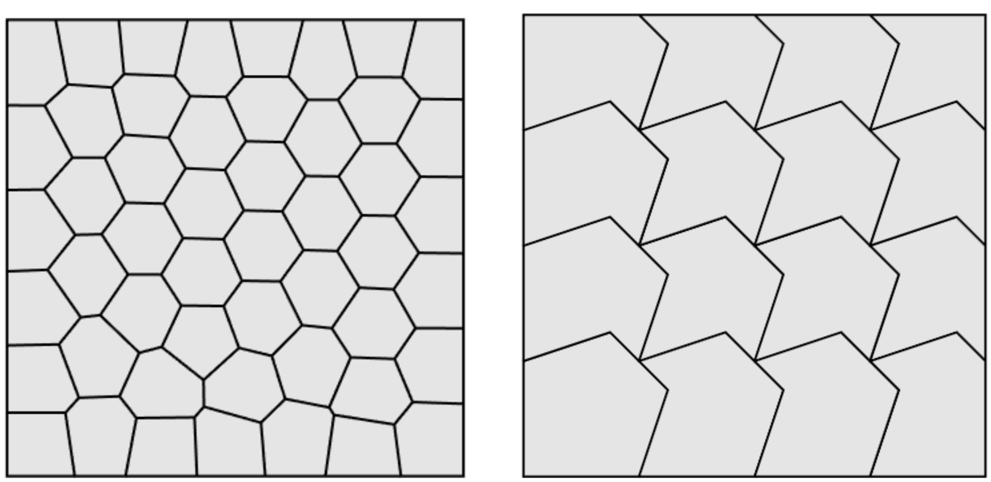
Substituting the estimates (4.4.29) and (4.4.30), we obtain,

$$|||u - u_h|||^2 \leq C\,[|||u - u_I|||^2 + |||u_I - u_h|||^2] \leq C\,h^{2s}(\epsilon + h). \tag{4.4.31}$$

□

4.5 Numerical Experiments

In this section, we perform numerical experiments to validate our theoretical convergence estimate derived in Theorem 4.2. As the model problem is nonlinear, the discrete scheme (4.2.13) results in nonlinear system of equations. In the first experiment, we solve this system using Newton's method and show the obtained rate of convergence using convergence plots. However, the numerical solution obtained using this approach is time consuming. In order to improve the time efficiency, that is, to reduce CPU time taken to solve the system, we perform the two-grid approach (see [79]) and compare the performance of both these techniques in our second numerical test. In the two-grid method, we consider two VEM spaces V_h^p and V_h^p with mesh diameter $h < H$. At first, we obtain a discrete solution u_H of (4.2.13) in the coarse space V_h^p using the standard Newton's method. Then, at the finer space V_h^p, we incorporate u_H into the discrete scheme and perform only two Newton's iterations, to obtain the solution u_h. In order to obtain the optimal accuracy, we consider $h \leq H^2$.



(a) (b)

Figure 4.1: Representative Voronoi and non-convex mesh employed in this study.

We have considered two types of meshes namely regular Voronoi and non-convex mesh on a square domain for the considered numerical tests. For step size $h = 1/5$, we have shown the sample meshes in Figure 4.1.
Let u, u_h represent the exact and discrete solution, respectively. In the examples, we evaluate the $H^1(\Omega)$ semi-norm and the energy norm denoted by $e_{h,1}$ and $e_{h,|||\cdot|||}$, defined as follows,

$$\begin{aligned} e_{h,1}^2 &= \sum_{E\in\mathcal{T}_h} \|\nabla(u - \Pi_p^\nabla u_h)\|_E^2, \\ e_{h,|||\cdot|||}^2 &= \sum_{E\in\mathcal{T}_h} \left(\|\sqrt{K}\nabla(u - \Pi_p^\nabla u_h)\|_E^2 + \tau_E\|\boldsymbol{\beta}(\Pi_p^0 u_h)\cdot\nabla(u - \Pi_p^\nabla u_h)\|_E^2\right). \end{aligned}$$

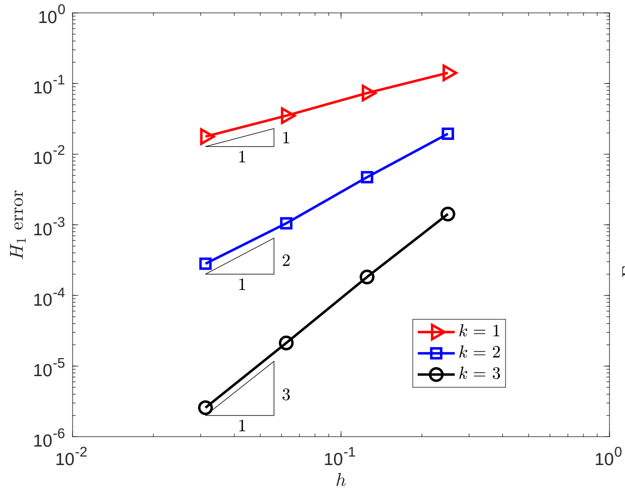
4.5.1 Example 1

Consider the unit square domain $\Omega = [0,1]\times[0,1]$ and choose the exact solution $u(x,y) := x\,y\sin(\pi x)\sin(\pi y)$. The coefficients are taken as $\epsilon = 10^{-6}$, $\boldsymbol{\beta}(u) = (u,u)^T$ and $r(u) = 2\,u + f$, where function f is defined such that u satisfies (4.1.1).
The convergence graphs are shown in Figure 4.2 for the H^1 semi-norm and energy norm, and for VEM orders 1,2 and 3,respectively. As predicted in Theorem 4.2, we observe that the method converges numerically to the expected rate of convergence.

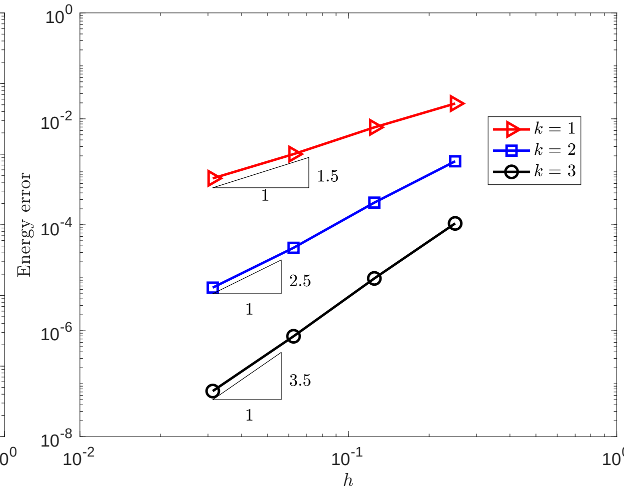
4.5.2 Example 2

Consider the unit square domain $\Omega = [0,1]\times[0,1]$ and choose the exact solution $u(x,y) := e^2\,x\,(x-1)^2\,y\,(y-1)^2$. The coefficients are taken as $\epsilon = 10^{-6}$, $\boldsymbol{\beta}(u) = (u,u)^T$ and $r(u) = 5\,u + f$, where function f is determined such that u satisfies (4.1.1).

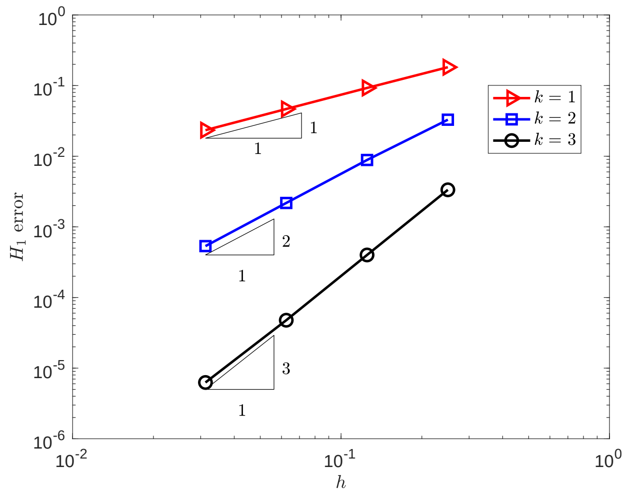
As mentioned in the numerical setting (Section 4.5), in order to reduce the computational cost involved in solving the nonlinear system of equations, we have used two-grid approach. Table 4.1 and Table 4.2 shows the comparison between Newton's method and two-grid approach for Voronoi mesh for the VEM order $k = 1$ and $k = 2$, respectively. We observe from the tables that CPU time of the two-grid method is halved compared to the Newton's method when the mesh diameters are decreased. Tables 4.3 and 4.4, shows the CPU time comparison for the non-convex mesh. Similar to Voronoi mesh case, two-grid approach takes lesser time than the Newton's method on a single grid.



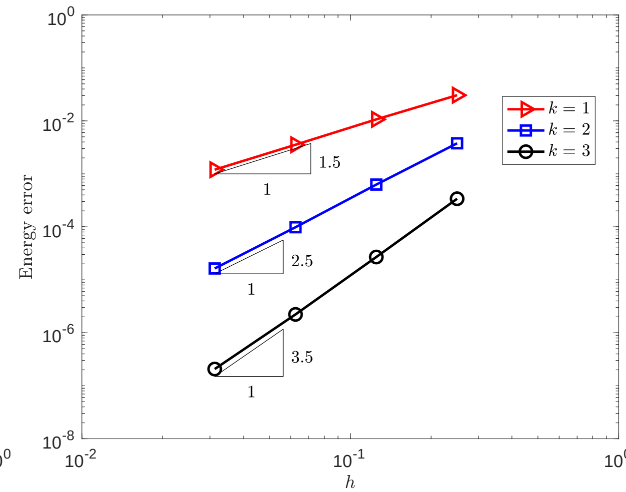
(a) $e_{h,1}$ errors



(b) $e_{h,|||\cdot|||}$ errors



(c) $e_{h,1}$ errors



(d) $e_{h,|||\cdot|||}$ errors

Figure 4.2: Rate of convergence plot in the H^1 semi-norm and energy norm for (a)-(b) Voronoi mesh and (c)-(d) Non-convex mesh for VEM orders $k = 1$, 2 and 3.

	Newton's method			Two-grid method			
h	$e_{h,\|\|\|\cdot\|\|\|}$	rate	Time	H	$e_{h,\|\|\|\cdot\|\|\|}$	rate	Time
$1/8$	$1.689365e^{-3}$	-	2.99	$1/4$	$1.536699e^{-3}$	-	2.38
$1/16$	$7.071470e^{-4}$	1.25	12.71	$1/4$	$5.071543e^{-4}$	1.59	6.41
$1/32$	$2.756350e^{-4}$	1.36	51.39	$1/8$	$1.835257e^{-4}$	1.47	25.01

Table 4.1: CPU time comparison: Newton's method and two-grid method for the VEM order $k = 1$ using Voronoi mesh.

h	Newton's method $e_{h,\|\|\|\cdot\|\|\|}$	rate	Time	H	Two-grid method $e_{h,\|\|\|\cdot\|\|\|}$	rate	Time
$1/8$	$9.361893e^{-5}$	-	2.56	$1/4$	$9.361952e^{-5}$	-	2.24
$1/16$	$1.329525e^{-5}$	2.82	14.05	$1/4$	$1.329529e^{-5}$	2.83	6.81
$1/32$	$2.485511e^{-6}$	2.41	112.70	$1/8$	$2.485511e^{-6}$	2.42	55.11

Table 4.2: CPU time comparison: Newton's method and two-grid method for the VEM order $k = 2$ using Voronoi mesh.

h	Newton method $e_{h,\|\|\|\cdot\|\|\|}$	rate	Time	H	Two-grid method $e_{h,\|\|\|\cdot\|\|\|}$	rate	Time
$1/8$	$2.341559e^{-3}$	-	2.33	$1/4$	$2.223864e^{-3}$	-	1.41
$1/16$	$1.052441e^{-3}$	1.15	5.44	$1/4$	$7.758496e^{-4}$	1.52	4.42
$1/32$	$4.166264e^{-4}$	1.34	41.39	$1/8$	$2.650724e^{-4}$	1.55	19.74

Table 4.3: CPU time comparison: Newton's method and two-grid method for the VEM order $k = 1$ using non-convex mesh.

h	Newton method $e_{h,\|\|\|\cdot\|\|\|}$	rate	Time	H	Two-grid method $e_{h,\|\|\|\cdot\|\|\|}$	rate	Time
$1/8$	$2.260889e^{-4}$	-	1.95	$1/4$	$2.260918e^{-4}$	-	1.33
$1/16$	$4.089832e^{-5}$	2.46	9.89	$1/4$	$4.090097e^{-5}$	2.47	4.95
$1/32$	$6.897373e^{-6}$	2.57	75.09	$1/8$	$6.897400e^{-6}$	2.57	37.40

Table 4.4: CPU time comparison: Newton's method and two-grid method for the VEM order $k = 2$ using non-convex mesh.

4.6 Summary

In this article, we have analysed the SUPG stabilized Virtual element method for quasi-linear convection-diffusion-reaction equation. We have used suitable polynomial projection operators and VEM stabilizers with appropriate coefficients, in the formulation of the discrete scheme. This ensures computablity and stability of the VEM discretisation. Most importantly, we have proved the existence and uniqueness of discrete solutions approximating the branch of solutions. We also proved the convergence estimate by showing the optimal rate of convergence in the energy norm and H^1 seminorm. We conducted numerical experiments using higher order virtual element method of orders $p = 1, 2, 3$.

Numerical simulation of the nonlinear problem over a fine grid is always time-consuming and thus computationally expensive. In order to address this issue, we use two-grid method which solves the nonlinear equations on two grids of different sizes, which significantly reduces the time complexity. We have performed the numerical experiments with the two-grid method and compared it over the standard Newton iterative approach. We observed from the tabulated results that CPU time of the two-grid method is halved compared to the Newton's method, for very fine mesh that is, when the mesh diameters are decreased. Also, we note that the two-grid performs efficiently without compromising on the accuracy, independent of the type of mesh.

Chapter 5

Virtual Element Analysis of Nonlocal Coupled Time-dependent Reaction-Diffusion Equations on Polygonal Meshes

We present a virtual element framework for the nonlocal coupled time-dependent reaction-diffusion problem. Such problems find nitsche applications in many fields of applied science and engineering, for example in modelling epidemics [83, 84], polymerization [85], tumor growth modeling [86], to name a few. The nonlocal coupled time-dependent reaction-diffusion problems belongs to a wider class of nonlinear problems, namely the nonlocal coupled parabolic problems. Henceforth, we shall address this problem with the latter terminology. In [87], the authors proved the existence and the uniqueness of the analytical solution of the nonlocal coupled parabolic problem. Numerical solutions based on the finite element method (FEM) and the virtual element method have been attempted in [88, 89]. In [88], author employed the conforming linear finite element method for the discretization of the non-local coupled parabolic problems.

In contrast to the FEM, the direct discretization of the nonlocal term will not be computable. Using the projection operator, the nonlocal term is discretized which is computable from the degrees of freedom related to the virtual element space. However, the presence of nonlocal coefficients in the system not only makes the computation of the Jacobian more expensive in Newton's method, but also destroys the sparsity structure of the Jacobian, consequently causing memory constrains and slowing of data processing, for large degrees of freedom. Following [90], an analogous approach is employed to rewrite the nonlinear system, such that the sparsity of the Jacobian is retained. Moreover, a linearized scheme for the coupled nonlocal parabolic problem is introduced that yields optimal order of convergence in both the space and the time variables. The nonlocal coefficients and the load terms can be computed from the previous steps and hence the fully discrete system reduces to a system of linear equations which can be computed easily.

5.0.1 Notations

Consider a convex polygonal domain $\Omega \subset \mathbb{R}^d$ where $d = 2, 3$ represents the dimension of the domain, with Lipschitz boundary $\partial\Omega$. We define the final time T and the time interval $I = [0, T]$. Further, we denote $L^2(\Omega)$, the space of square integrable functions with standard inner-product $(\phi, \psi)_\Omega := \int_\Omega \phi\, \psi\, d\Omega$. For each positive integer $s \in \mathbb{N}$, we define $H^s(\Omega)$, the Sobolev space with standard norm $\|\phi\|_s := \Big(\sum\limits_{0 \le \alpha \le s} \|D^\alpha \phi\|^2 \Big)^{1/2}$, where $D^\alpha \phi$ denotes α^{th} partial derivative of ϕ. Moreover, the function space $L^2(0, T; H^s(\Omega))$ consists of function ϕ such that for almost all $t \in [0, T]$, $\phi(\cdot, t) \in H^s(\Omega)$ with the norm

$$\|\phi\|_{L^2(0,T,H^s(\Omega))} := \Big(\int_0^T \|\phi(t)\|_s^2 \Big)^{1/2}; \quad \|\phi\|_{L^\infty(0,T,H^s(\Omega))} = \operatorname*{ess\,sup}_{0 \le t \le T} \|\phi(t)\|_s.$$

In addition, we define $\mathbb{P}_k(E)$, the space of all polynomials of degree less than or equal to k on E and for a function v, the first and the double derivatives with respect to t are denoted by $D_t v$, $D_{tt} v$, respectively.

5.1 The continuous problem

Let $f_i(u, v) \in L^2(\Omega, I)$ be the force functions for $i \in \{1, 2\}$, and u_0, and v_0 be the initial guess for the solutions $u,\ v$, respectively. The continuous problem is then given by: find (u, v) such that for $t \in [0, T]$, we have:

$$D_t u - \mathcal{A}_1(g_1(u), g_2(v))\, \Delta u = f_1(u, v) \quad \text{in} \quad \Omega \times (0, T), \tag{5.1.1}$$

$$D_t v - \mathcal{A}_2(g_1(u), g_2(v))\, \Delta v = f_2(u, v) \quad \text{in} \quad \Omega \times (0, T), \tag{5.1.2}$$

$$u(x, t) = v(x, t) = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{5.1.3}$$

$$u(x, 0) = u_0(x) \quad \text{on} \quad \Omega, \tag{5.1.4}$$

$$v(x, 0) = v_0(x) \quad \text{on} \quad \Omega, \tag{5.1.5}$$

where $g_i(\omega) := \displaystyle\int_\Omega l_i(x)\, \omega\, d\Omega$ for $\omega(\cdot, t) \in L^2(\Omega)$ for almost all $t \in [0, T]$ and $l_i(x) \in L^2(\Omega)$. Since the diffusive coefficients $\mathcal{A}_i' s$ are dependent on the global behaviour of the solution, the problem is termed nonlocal.

Further, we will make some assumptions on the model problem in order to derive the theoretical estimates in the later section.

Assumption 5.1.

- For $i \in \{1,2\},\ \mathcal{A}_i(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is bounded i.e., $0 < m_0 < \mathcal{A}_i(\cdot,\cdot) < M$, where m_0 and M are positive constants.

- $\mathcal{A}_i(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous i.e

$$|\mathcal{A}_i(r_1, s_1) - \mathcal{A}_i(r_2, s_2)| \leq\ L_A(|r_1 - r_2| + |s_1 - s_2|) \quad \forall (r_i, s_i) \in \mathbb{R} \times \mathbb{R}. \tag{5.1.6}$$

- For $i \in \{1,2\}$, the right hand side force function, f_i are Lipschitz continuous w.r.t. u and v. i.e.,

$$|f_i(u_1, v_1) - f_i(u_2, v_2)| \leq L_F(|u_1 - u_2| + |v_1 - v_2|) \quad \forall u_1, u_2, v_1, v_2 \in \mathbb{R}. \tag{5.1.7}$$

Multiplying equation (5.1.1) by the test function φ and (5.1.2) by test function ψ and employing divergence theorem, we derive the continuous weak formulation: Find $u,\ v \in L^2(0,T; H_0^1(\Omega) \cap C(0,T; L^2(\Omega))$ and $D_t u,\ D_t v \in L^2(0,T; H^{-1}(\Omega))$ for almost all $t \in [0,T]$ such that

$$\frac{d}{dt}(u, \varphi) + \mathcal{A}_1(g_1(u), g_2(v))(\nabla u, \nabla \varphi) = \langle f_1(u,v), \varphi \rangle \text{ in } \mathcal{D}'(0,T)\ \forall \varphi \in V = H_0^1(\Omega), \tag{5.1.8}$$

$$\frac{d}{dt}(v, \psi) + \mathcal{A}_2(g_1(u), g_2(v))(\nabla v, \nabla \psi) = \langle f_2(u,v), \psi \rangle \text{ in } \mathcal{D}'(0,T)\ \forall \psi \in V = H_0^1(\Omega), \tag{5.1.9}$$

$$u(\mathbf{x}, t) = v(\mathbf{x}, t) = 0 \quad \text{for} \quad (\mathbf{x}, t) \in\ \partial\Omega \times (0,T), \tag{5.1.10}$$

$$u(\mathbf{x}, t) = u_0(\mathbf{x}) \quad \text{and} \quad v(\mathbf{x}, t) = v_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega, \tag{5.1.11}$$

where $\mathcal{D}'(0,T)$ is the space of distributions on (0,T) and $\langle \cdot, \cdot \rangle$ denotes the $V'-$ duality bracket. The existence and the uniqueness of the weak solution satisfying equation (5.1.8) to (5.1.11) can be easily proved using Brouwer's fixed point arguments [91].

Theorem 5.1. *Let the assumption 5.1 hold. Then, there exists a unique solution* $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *of the problem* (5.1.8) - (5.1.11).

Using the assumption 2, Schauder's fixed point theorem and proceeding analogously as in [91, Theorem 2.1], we get the desired result.

5.2 Virtual Element Methods

In this section, we consider the necessary assumptions on the mesh elements and discuss the construction of two and three dimensional virtual element space which were originally introduced in [26]. Unlike the finite element space, the virtual element space consists of both polynomial functions and some implicitly defined non-polynomial functions. To tackle the non-polynomial functions in the evaluation of bilinear forms, we use suitable polynomial projection operators on the functions of the virtual element space that ensures computability using only the known degrees of freedom (DoFs).

5.2.1 Mesh Regularity

Let $\{\Sigma_h\}_h$ be a sequence of polytopal meshes consisting of polygonal/polyhedral elements E or P and let h_E/h_P be the diameter of an element $E/P \in \Sigma_h$; $h := \max\limits_{E\in\Sigma_h} h_E$ and for polyhedron, $h := \max\limits_{P\in\Sigma_h} h_P$. In continuation, we define $e/F \subset \partial E/\partial P$ be an arbitrary edge/face and $\partial E/\partial P$ be the boundary of E/P. Moreover, we consider the following regularity conditions on the domain decomposition.

Assumption 5.2.

(T_1) $E \in \Sigma_h$ is star shaped with respect to every point of a ball of radius greater than $\gamma\, h_E$.

(T_2) for every element E, and for every $e \subset \partial E$ satisfies $h_e > h_E$.

(T_3) for polyhedral elements $P \subset \mathbb{R}^3$, each face $F \subset \partial P$ satisfies (T_1) and (T_2).

where $\gamma > 0$ is a positive constant.

The following canonical convention of the multi-dimensional space is followed. Let $\mathbf{s} = (s_1, s_2, \ldots, s_d)$ and define $|\mathbf{s}| = s_1 + s_2 + \cdots + s_d$. We denote a element $\mathbf{x}^{\mathbf{s}} \in \mathbb{R}^d,\ d = 2, 3$ by, $\mathbf{x}^{\mathbf{s}} := (\, x_1^{s_1}\ x_2^{s_2}\ \ldots x_d^{s_d}\,)$, and $\mathbf{x}_E$ be the centroid of polygon E. In what follows, $\mathcal{M}_k^d(E) := \left\{ \left(\frac{\mathbf{x}-\mathbf{x}_E}{h_E}\right)^{s}, |s| \le k \right\}$, $d = 2, 3$ is the set of scaled monomials with $\mathcal{M}_{-1}^d(E) = \{0\}$.

Consider the L^2 projection operator $\Pi^0_{k,E} : L^2(E) \to \mathbb{P}_k(E)$ defined such that

$$\big((\Pi^0_{k,E} - I)u,\, v \big)_E = 0 \qquad \forall v \in \mathbb{P}_k(E),$$

and define the elliptic projection operator $\Pi^\nabla_{k,E} : H^1(E) \to \mathbb{P}_k(E)$ satisfying,

$$\big(\nabla(\Pi^\nabla_{k,E} - I)u,\, \nabla v \big)_E = 0 \quad \forall v \in \mathbb{P}_k(E) \quad \text{and} \quad \int_{\partial E} (\Pi^\nabla_{k,E} u - u)\, dr = 0.$$

5.2.1.0.1 Two dimensional virtual element space

For every $E \in \mathcal{T}_h$, consider the space W_E^p (see [26]) defined by,

$$W_E^k = \left\{ v \in H^1(E) \cap C^0(\partial E) : v_{|e} \in \mathbb{P}_k(e) \, \forall \, \text{edge} \, e \in \partial E, \Delta v \in \mathbb{P}_k(E) \right\}.$$

Next, we introduce the local virtual element space in two dimensions. Let

$$\mathcal{H}^k(E) := \left\{ v \in W_E^k \ : \ \int_E (\Pi^{\nabla}_{k,E} v - v) \, q = 0 \ \forall \, q \in \mathbb{P}_k \setminus \mathbb{P}_{k-2}(E) \right\}, \tag{5.2.1}$$

where $\mathbb{P}_k \setminus \mathbb{P}_{k-2}(E)$ denotes the set of polynomials of degrees exactly equal to $k-1$ and k.

Now we define a set of DOFs associated with an element $u \in \mathcal{H}^k(E)$:

- The values of u at the vertices of the element E.
- On each edge $e \subset \partial E$, the moments of u up to order $k-2$ i.e.

$$\frac{1}{|e|} \int_e u \, \omega \, \mathrm{de} \quad \forall \omega \in \mathcal{M}^1_{k-2}(e).$$

- The moments up to order $k-2$ of u in E i.e.,

$$\frac{1}{|E|} \int_E u \, \omega \, \mathrm{dE}, \qquad \forall \omega \in \mathcal{M}^2_{\mathrm{k}-2}(\mathrm{E}),$$

We see that the above set of DOFs are unisolvent (see [27, 30, 92]).

Then the two dimensional global virtual element space is defined as follows

$$\mathcal{H}_h^k := \{ v \in H_0^1(\Omega) \, \big| \, v|_E \in \mathcal{H}^k(E) \ \forall E \in \Sigma_h \}. \tag{5.2.2}$$

5.2.1.0.2 Three dimension virtual element space

The construction of the conforming virtual element space for $d = 3$ follows an analogous idea as $d = 2$. For each polyhedral element P, we define

$$\mathcal{B}_k^3(\partial P) := \{ v \in C^0(\partial P) : v|_F \in \mathcal{H}^k(F) \, \forall F \subset \partial P \},$$

where $\mathcal{H}^k(F)$ is the conforming two dimensional local virtual element space of degree k over face F. Following [27, 93], the auxiliary space is defined as

$$\mathcal{H}^k(P) := \Big\{ w \in H^1(P) \, : w|_{\partial P} \in \mathcal{B}_k^3(\partial P), \;\; \Delta w \in \mathbb{P}_k(P); \\ \text{and } (w - \Pi_{k,P}^{\nabla} w, q)_P = 0 \;\; \forall q \in \mathbb{P}_k \setminus \mathbb{P}_{k-2}(P) \Big\},$$

where the operator $\Pi_{k,P}^{\nabla}$ is elliptic projection operator on polyhedral element P. Further, we define the DOFs associated with a function u in the virtual space $\mathcal{H}^k(P)$ (see [30, 92, 93]):

- The values of u at the vertices of the element P.
- On each edge $e \subset \partial P$, the moment of the function u up to order $k-2$ i.e.

$$\frac{1}{|e|} \int_e u \, \omega \, \mathrm{de} \quad \forall \omega \in \mathcal{M}^1_{\mathrm{k-2}}(\mathrm{e}).$$

- The moments up to order $k-2$ of u on each face $F \subset \partial P$.

$$\frac{1}{|F|} \int_F u \, \omega \, \mathrm{dF}, \quad \forall \omega \in \mathcal{M}^2_{\mathrm{k-2}}(\mathrm{F}).$$

- The moments up to order $k-2$ of u in P i.e.,

$$\frac{1}{|P|} \int_P u \, \omega \, \mathrm{dP}, \qquad \forall \, \omega \in \mathcal{M}^3_{\mathrm{k-2}}(\mathrm{P}).$$

Finally, the global conforming virtual element space is defined as:

$$\mathcal{H}_h^k := \{ v \in H_0^1(\Omega) : v|_P \in \mathcal{H}^k(P) \; \forall P \in \Sigma_h \}.$$

Hereafter, we will not make any difference between E and P.

Remark 5.1. It can be observed that the local virtual element space $\mathcal{H}^k(E)$ has the same number of DoFs as [26] with an added advantage that the L^2 projection operator $\Pi^0_{k,E}$ is computable on $\mathcal{H}^k(E)$ [27]. The L^2 projection operator is used to discretize the nonlocal term and the non-stationary part of the model problem that will be discussed in the later sections.

Next, we introduce the discrete bilinear form $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$ corresponding to the continuous form $a(\cdot,\cdot)$ and $m(\cdot,\cdot)$ respectively. Since, the virtual space contains polynomial and non-polynomial functions that are not available in a closed form, we employ the projection operators, $\Pi^0_{k,E}$ and $\Pi^{\nabla}_{k,E}$ to discretize the bilinear form.

First let us consider the symmetric bilinear forms $S^E_a(\cdot,\cdot)$ and $S^E_m(\cdot,\cdot)$ that are positive semi-definite and definite, respectively, on $\mathcal{H}^k_h \times \mathcal{H}^k_h$ and are such that, there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$\begin{aligned} \alpha_1 \, a^E(v,v) \leq S^E_a(v,v) \leq \alpha_2 \, a^E(v,v) \quad \forall v \in \mathcal{H}^k(E) \cap \mathrm{Ker}(\Pi^{\nabla}_{k,E}) \\ \beta_1 \, (w,w)_E \leq S^E_m(w,w) \leq \beta_2 \, (w,w)_E \quad \forall w \in \mathcal{H}^k(E) \cap \mathrm{Ker}(\Pi^0_{k,E}). \end{aligned}$$

It can be observed that $S^E_a(\cdot,\cdot)$ or $S^E_m(\cdot,\cdot)$ reduce to zero when one of the entries is a polynomial. Then the local bilinear form $a^E_h(\cdot,\cdot) : \mathcal{H}^k(E) \times \mathcal{H}^k(E) \to \mathbb{R}$ and $m_h(\cdot,\cdot) : \mathcal{H}^k(E) \times \mathcal{H}^k(E) \to \mathbb{R}$ are defined as follows:

$$\begin{aligned} a^E_h(w,v) &:= a^E(\Pi^{\nabla}_{k,E} w, \Pi^{\nabla}_{k,E} v) + S^E_a((I-\Pi^{\nabla}_{k,E})w, (I-\Pi^{\nabla}_{k,E})v) \quad \forall w,v \in \mathcal{H}^k(E), \\ m^E_h(w,v) &:= (\Pi^0_{k,E} w, \Pi^0_{k,E} v)_E + S^E_m((I-\Pi^0_{k,E})w, (I-\Pi^0_{k,E})v) \quad \forall w,v \in \mathcal{H}^k(E). \end{aligned} \tag{5.2.3}$$

Amongst the different computable forms of the projection operators available in the literature [28], we choose the following representation:

$$S^E_m(\phi,\psi) := \; h^d_E \sum_{z=1}^{N^{\mathrm{dof}}_E} \mathrm{dof}_z(\phi) \, \mathrm{dof}_z(\psi), \quad \text{and} \quad S^E_a(\phi,\psi) := \; h^{d-2}_E \sum_{z=1}^{N^{\mathrm{dof}}_E} \mathrm{dof}_z(\phi) \, \mathrm{dof}_z(\psi),$$

where d is the dimension of the space, and N^{dof}_E denotes dimension of the local space $\mathcal{H}^k(E)$. The local forms $a^E_h(\cdot,\cdot)$ and $m^E_h(\cdot,\cdot)$ satisfy the following two properties :

Polynomial consistency: For an element $E \in \Sigma_h$, $0 < h \leq 1$, the bilinear forms $a^E_h(\cdot,\cdot)$ and $m^E_h(\cdot,\cdot)$ defined in (5.2.3), satisfy the following consistency properties:

$$\begin{aligned} a^E_h(p,v) = a^E(p,v) \quad \forall p \in \mathbb{P}_k(E), \quad \forall v \in \mathcal{H}^k(E) \\ m^E_h(p,v) = (p,v)_E \quad \forall p \in \mathbb{P}_k(E), \quad \forall v \in \mathcal{H}^k(E). \end{aligned} \tag{5.2.4}$$

Stability: There exist four mesh independent positive constants, $\alpha^*, \alpha_*, \beta^*, \beta_*$ independent of the element E such that for all $v \in \mathcal{H}^k(E)$, $a^E_h(v,v)$, and $m^E_h(v,v)$ are bounded by $a^E(v,v)$ and $(v,v)_E$, respectively, i.e.,

$$\begin{aligned} \alpha_* \, a^E(v,v) \leq a^E_h(v,v) \leq \alpha^* \, a^E(v,v); \\ \beta_*(v,v)_E \leq m^E_h(v,v) \leq \; \beta^*(v,v)_E \end{aligned} \tag{5.2.5}$$

hold. Condition (5.2.5) ensures that the non-polynomial parts $S_a^E(\cdot,\cdot)$ and $S_m^E(\cdot,\cdot)$ scale same as polynomial parts of $a_h^E(\cdot,\cdot)$ and $m_h^E(\cdot,\cdot)$, respectively.

Adding the local contributions, the global form $a_h(\cdot,\cdot) : \mathcal{H}_h^k \times \mathcal{H}_h^k \to \mathbb{R}$ and $m_h(\cdot,\cdot) : \mathcal{H}_h^k \times \mathcal{H}_h^k \to \mathbb{R}$ are defined as

$$a_h(w,v) := \sum_{E\in\Sigma_h} a_h^E(w,v) \quad \text{and} \quad m_h(w,v) := \sum_{E\in\Sigma_h} m_h^E(w,v) \qquad \forall w,v \in \mathcal{H}_h^k.$$

Remark 5.2. To discretize the bilinear form $a^E(\cdot,\cdot)$, we have employed $\Pi_{k,E}^{\nabla}$ operator. However, the term $a^E(\cdot,\cdot)$ can be discretized by employing the external projection operator $\Pi_{k-1,E}^0$ [28].

Remark 5.3. In this work, we use the projection operators' matrix representation to evaluate the matrices corresponding to the bilinear forms $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$ respectively. This matrix representation depends on the order of the space and shape of the element E, but is independent of the size of the element. Therefore, the matrices remains unchanged for any transformations that preserves the shape of E. However, this inspection is not true for higher order virtual element space. [29, Remark 3.5]. We compute the matrices following the procedure highlighted in [29].

5.2.2 Semi-discrete formulation

Using the discrete bilinear form, the semi discrete formulation of (5.1.8)-(5.1.11) is defined as: find $(u_h(t),\, v_h(t)) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ for all most all $t \in [0,T]$ such that

$$m_h(D_t u_h, \varphi_h) + \mathcal{A}_1(g_1(\Pi_k^0 u_h), g_2(\Pi_k^0 v_h))\, a_h(u_h, \varphi_h) = \langle f_{1h}(u_h, v_h), \varphi_h\rangle \quad \forall \varphi_h \in \mathcal{H}_h^k, \tag{5.2.6}$$

$$m_h(D_t v_h, \psi_h) + \mathcal{A}_2(g_1(\Pi_k^0 u_h), g_2(\Pi_k^0 v_h))\, a_h(v_h, \psi_h) = \langle f_{2h}(u_h, v_h), \psi_h\rangle \quad \forall \psi_h \in \mathcal{H}_h^k, \tag{5.2.7}$$

where

$$\begin{aligned} \langle f_{1h}(u_h, v_h), \varphi_h\rangle &= \sum_{E\in\Sigma_h} \int_E f_1(\Pi_{k,E}^0 u_h,\, \Pi_{k,E}^0 v_h)\, \Pi_{k,E}^0 \varphi_h \, dE, \\ \text{and} \quad \langle f_{2h}(u_h, v_h),\, \psi_h\rangle &= \sum_{E\in\Sigma_h} \int_E f_2(\Pi_{k,E}^0 u_h,\, \Pi_{k,E}^0 v_h)\, \Pi_{k,E}^0 \psi_h \, dE. \end{aligned} \tag{5.2.8}$$

The scheme (5.2.6)-(5.2.7) constitute a system of differential equations. Since the model problem satisfies assumptions 5.1, we deduce that the nonlinear system of equations (5.2.6)-(5.2.7) has a unique solution for $t \in [0, T_1]$, where $T_1 < T$. Such a solution can be extended to $[0, T]$ following the boundedness property of the discrete solutions. Let C be a generic positive constant that is independent of mesh diameter h and E, which takes different values at different instances.

Theorem 5.2. *Let the discrete solutions* $(u_h^0, v_h^0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and the two force functions* $f_1(u,v), f_2(u,v) \in L^2(0,T,L^2(\Omega))$*, then, the solution of* (5.2.6)-(5.2.7) (u_h, v_h) *satisfies the following boundedness property*

$$\begin{aligned} &\|v_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad \|u_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \\ &\|D_t v_h\|_{L^2(0,T;L^2(\Omega))} \leq C \quad \|D_t u_h\|_{L^2(0,T;L^2(\Omega))} \leq C. \end{aligned}$$

Proof. We consider the semi-discrete formulation (5.2.6)-(5.2.7). Upon choosing $\varphi_h = u_h$ in (5.2.6), we obtain

$$\frac{1}{2}\frac{d}{dt} m_h(u_h, u_h) + \mathcal{A}_1(g_1(\Pi_k^0 u_h), g_2(\Pi_k^0 v_h))\, a_h(u_h, u_h) = \langle f_{1h}(u_h, v_h), u_h \rangle \tag{5.2.9}$$

Using assumption 5.1, triangle inequality and continuity of operator Π_k^0, we can get

$$\begin{aligned} \|f_1(\Pi_k^0 u_h, \Pi_k^0 v_h)\|_0 &= \|f_1(\Pi_k^0 u_h, \Pi_k^0 v_h) - f_1(0,0) + f_1(0,0)\|_0 \\ &\leq L_F\,(\,\|u_h\|_0 + \|v_h\|_0\,) + \|f_1(0,0)\|_0. \end{aligned} \tag{5.2.10}$$

An application of Cauchy-Schwarz inequality, boundedness of operator Π_k^0, Young's inequality and (5.2.10), we obtain

$$|\,\langle f_{1h}(u_h, v_h), u_h \rangle\,| \leq \frac{1}{2}(\|f_1(\Pi_k^0 u_h, \Pi_k^0 v_h)\|_0^2 + \|u_h\|_0^2) \leq C\Big(\|u_h\|_0^2 + \|v_h\|_0^2 + \|f_1(0,0)\|_0^2\Big). \tag{5.2.11}$$

Substituting the estimation (5.2.11) into (5.2.9), we have

$$\frac{1}{2}\,\beta_*\,\frac{d}{dt}\,\|u_h\|_0^2 + m_0\,\alpha_*\|\nabla u_h\|_0^2 \leq C\Big(\|u_h\|_0^2 + \|v_h\|_0^2 + \|f_1(0,0)\|_0^2\Big). \tag{5.2.12}$$

In the analogous way, we obtain

$$\frac{1}{2}\beta_*\frac{d}{dt}\|v_h\|_0^2 + m_0\,\alpha_*\|\nabla v_h\|_0^2 \leq C\Big(\|u_h\|_0^2 + \|v_h\|_0^2 + \|f_2(0,0)\|^2\Big). \tag{5.2.13}$$

Adding (5.2.12) and (5.2.13), we have

$$\begin{aligned}\frac{1}{2}\,\beta_*\,\frac{d}{dt}(\|u_h\|_0^2+\|v_h\|_0^2)+m_0\,\alpha_*(\|\nabla u_h\|_0^2+\|\nabla v_h\|_0^2)&\\ \leq C\Big(\|u_h\|_0^2+\|v_h\|_0^2+\|f_1(0,0)\|_0^2+\|f_2(0,0)\|_0^2\Big).&\end{aligned} \tag{5.2.14}$$

Integrating both sides of (5.2.14), and on application of Grownwall's inequality, we obtain:

$$\begin{aligned}(\|u_h(t)\|^2+\|v_h(t)\|^2)+C(m_0,\beta_*,\alpha_*)\,\int_0^t\big(\,\|\nabla u_h\|^2+\|\nabla v_h\|^2\,\big)dr&\\ \leq C\Big(\,\|u_h^0\|_0^2+\|v_h^0\|_0^2+\|f_1(0,0)\|_0^2+\|f_2(0,0)\|_0^2\,\Big).&\end{aligned} \tag{5.2.15}$$

for all $t\in[0,T]$ which implies that $\|u_h\|_{L^\infty(0,T:L^2(\Omega))}$ and $\|v_h\|_{L^\infty(0,T:L^2(\Omega))}$ are bounded.

In order to bound the term $\|D_t u\|_{L^2(0,T;L^2(\Omega))}<\infty$ and $\|D_t v\|_{L^2(0,T;L^2(\Omega))}<\infty$, we choose $\varphi_h=D_t u_h$ in (5.2.6) and $\psi_h=D_t v_h$ in (5.2.7), and proceed as above similar to the line of proof of $\|u_h\|_{L^2(0,T:L^2(\Omega))}<\infty$ and $\|v_h\|_{L^2(0,T:L^2(\Omega))}<\infty$. □

5.2.3 Fully Discrete Scheme

We employ the virtual element method and backward Euler method for discretizing the space variable and the time variable, respectively. To this end we consider a partition of non-overlapping sub interval $[t_{n-1},t_n]$ of $[0,T]$, where $n=0,1,2,\cdots,N_T$ with time-step $\Delta t^n:=t_n-t_{n-1}$ such that $T=\sum_{n=0}^{N_T}\Delta t^n$. To reduce the computational complexity let $\Delta t^n=\Delta t$ for all n, i.e equal time steps. Let $\{(U^n,V^n)\}_{n\in\mathbb{N}}$ be a sequence of approximations of (u,v) at time $t=t_n$. Then the fully discrete scheme of (5.1.8)-(5.1.11) is defined as:

for each $n=1,2,3.....,N_T$, find $(U^n,\,V^n)\in\mathcal{H}_h^k\times\mathcal{H}_h^k$ such that

$$m_h\bigg(\frac{U^n-U^{n-1}}{\Delta t},\,\varphi_h\bigg)+\mathcal{A}_1(g_1(\Pi_k^0U^n),g_2(\Pi_k^0V^n))\,a_h(U^n,\varphi_h)=\langle f_{1h}(U^n,V^n),\varphi_h\rangle\,, \tag{5.2.16}$$

$$m_h\bigg(\frac{V^n-V^{n-1}}{\Delta t},\,\psi_h\bigg)+\mathcal{A}_2(g_1(\Pi_k^0U^n),g_2(\Pi_k^0V^n))\,a_h(V^n,\psi_h)=\langle f_{2h}(U^n,V^n),\psi_h\rangle\,. \tag{5.2.17}$$

$$U^0=I_h(u(t_0))\quad V^0=I_h(v(t_0)), \tag{5.2.18}$$

where U^0 and V^0 are initial approximation of u and v at time $t = 0$ respectively. The discrete scheme (5.2.16)-(5.2.17) reduces to a system of nonlinear equations which can be solved by employing iterative methods. To reduce the computation cost, we incorporate the technique introduced in [90]. A detailed implementation procedure will be discussed in subsection 5.2.5.

In addition to this, we would like to introduce a linearized scheme for the weak formulation (5.1.8)-(5.1.11). Here, when the unknowns are computed at time t_n, the nonlocal diffusive coefficients and the load terms are computed at the previous time-step t_{n-1}. We present the linearized scheme as follows:
for each $n = 1, 2, 3....., N_T$, find $(\widetilde{U}^n, \widetilde{V}^n) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ such that

$$m_h\left(\frac{\widetilde{U}^n - \widetilde{U}^{n-1}}{\Delta t}, \varphi_h\right) + \mathcal{A}_1(g_1(\Pi_k^0 \widetilde{U}^{n-1}), g_2(\Pi_k^0 \widetilde{V}^{n-1}))\, a_h(\widetilde{U}^n, \varphi_h) = \left\langle f_{1h}(\widetilde{U}^{n-1}, \widetilde{V}^{n-1}), \varphi_h \right\rangle \quad \forall \varphi_h \in \mathcal{H}_h^k, \tag{5.2.19}$$

$$m_h\left(\frac{\widetilde{V}^n - \widetilde{V}^{n-1}}{\Delta t}, \psi_h\right) + \mathcal{A}_2(g_1(\Pi_k^0 \widetilde{U}^{n-1}), g_2(\Pi_k^0 \widetilde{V}^{n-1}))\, a_h(\widetilde{V}^n, \psi_h) = \left\langle f_{2h}(\widetilde{U}^{n-1}, \widetilde{V}^{n-1}), \psi_h \right\rangle \quad \forall \psi_h \in \mathcal{H}_h^k \tag{5.2.20}$$

$$U^0 = I_h(u(t_0)) \quad V^0 = I_h(v(t_0)). \tag{5.2.21}$$

The discrete formulation (5.2.19)-(5.2.20) reduces to system of linear equations that can be solved by a linear solver directly. Let A and B be the matrix representation of the bilinear forms $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, which are positive semi-definite and positive definite, respectively. Further, let $\delta_u := \mathcal{A}_1(g_1(\Pi_k^0 \widetilde{U}^{n-1}), g_2(\Pi_k^0 \widetilde{V}^{n-1}))$ and $\delta_v := \mathcal{A}_2(g_1(\Pi_k^0 \widetilde{U}^{n-1}), g_2(\Pi_k^0 \widetilde{V}^{n-1}))$. Then, both the matrices $B + \Delta t \delta_u A$ and $B + \Delta t \delta_v A$ are invertible that ensures unique solutions to the system (5.2.19)-(5.2.21). Further, in Section 5.5, we will show that the approximation $(\widetilde{U}^n, \widetilde{V}^n)$ converges to the analytical solution with an optimal order in both the space and time variables. The rate of convergence depends on the initial approximation of the solution, i.e. (U^0, V^0). Therefore, the initial guess could be chosen as an interpolation of the analytical solution at $t = 0$.

5.2.4 Existence and uniqueness of the solution for the fully discrete scheme

Here, we shall use the following variant of Brouwer's fixed point theorem to ensure the existence of a solution for the discrete problem (5.2.16) - (5.2.18) and its uniqueness will be proved by the method of contradiction.

Theorem 5.3. *(Brouwer's Theorem) Let $\mathcal{K}$ be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{K}}$. Let $g:\mathcal{K}\to\mathcal{K}$ be a continuous function. If there exists a constant, $R>0$ such that $(g(z),z)_{\mathcal{K}}>0$ for all z with $\|z\|_{\mathcal{K}}=R$, then, there exists a $z^*\in\mathcal{K}$, such that $\|z^*\|_{\mathcal{K}}<R$ and $g(z^*)=0$.*

Remark 5.4. Let us define an inner product $[(X_1,Y_1),(X_2,Y_2)]:=(\nabla X_1,\nabla X_2)+(\nabla Y_1,\nabla Y_2)$. Then it is well-known that $\mathcal{H}_h^k\times\mathcal{H}_h^k$ is a finite dimensional Hilbert space with respect to $[\cdot,\cdot]$ and the induced norm $|||(X,Y)|||:=[(X,Y),(X,Y)]^{\frac{1}{2}}$.

Theorem 5.4. *Let $0\le n\le N_T$ and assume $(U^J,V^J)\in\mathcal{H}_h^k\times\mathcal{H}_h^k$ to be the given unique solution of the system* (5.2.16) - (5.2.18) *for $0\le J\le n-1$. Then for sufficiently small Δt, the system* (5.2.16)-(5.2.18) *has a unique solution $(U^n,V^n)\in\mathcal{H}_h^k\times\mathcal{H}_h^k$ at time t_n.*

Proof. We shall prove that the discrete system (5.2.16) -(5.2.18) has a solution (U^n,V^n) and that the solution is unique at $t=t_h$. For this reason we define a map

$$\mathcal{L}:\mathcal{H}_h^k\times\mathcal{H}_h^k\to\mathcal{H}_h^k\times\mathcal{H}_h^k \qquad \text{such that} \qquad \mathcal{L}(W_1,W_2):=(\,\mathcal{L}_1(W_1,W_2),\mathcal{L}_2(W_1,W_2)\,),$$

where $\mathcal{L}_i(W_1,W_2)\in\mathcal{H}_h^k$ for i=1,2 and satisfies,

$$\begin{aligned}(\,\nabla\mathcal{L}_1(W_1,W_2),\nabla\varphi\,) &:= m_h(W_1,\varphi_h)+\Delta t\mathcal{A}_1(g_1(\Pi_k^0W_1),g_2(\Pi_k^0W_2))\,a_h(W_1,\varphi_h)\\ &\quad-(\Delta t)\langle f_{1h}(W_1,W_2),\varphi_h\rangle-m_h(U^{n-1},\varphi_h)\qquad\forall\,\varphi\in\mathcal{H}_h^k.\end{aligned}\tag{5.2.22}$$

and

$$\begin{aligned}(\,\nabla\mathcal{L}_2(W_1,W_2),\nabla\psi\,) &:= m_h(W_2,\psi_h)+\Delta t\mathcal{A}_2(g_1(\Pi_k^0W_1),g_2(\Pi_k^0W_2))\,a_h(W_2,\psi_h)\\ &\quad-(\Delta t)\langle f_{2h}(W_1,W_2),\psi_h\rangle-m_h(V^{n-1},\psi_h)\quad\forall\,\psi_h\in\mathcal{H}_h^k.\end{aligned}\tag{5.2.23}$$

For each $(X,Y)\in\mathcal{H}_h^k\times\mathcal{H}_h^k$, let us define $\boldsymbol{T}_{X,Y}:\mathcal{H}_h^k\to\mathbb{R}$ by

$$\begin{aligned}\boldsymbol{T}_{X,Y}(\varphi_h) &:= m_h(X,\varphi_h)+\Delta t\,\mathcal{A}_1(g_1(\Pi_k^0X),g_2(\Pi_k^0Y))\,a_h(X,\varphi_h)\\ &\quad-(\Delta t)\,\langle f_{1h}(X,Y),\varphi_h\rangle-m_h(U^{n-1},\varphi_h).\end{aligned}$$

Note that for each $(X,Y)\in\mathcal{H}_h^k\times\mathcal{H}_h^k$, the corresponding $\boldsymbol{T}_{X,Y}$ is a bounded linear functional on $\mathcal{H}_h^k$ (follows, since the bilinear form $m_h(\cdot,\cdot)$, $a_h(\cdot,\cdot)$ are bounded, and the nonlocal term $\mathcal{A}_1(\cdot,\cdot)$, the force functions f_{1h} is Lipschitz continuous). Now, using Riesz representation theorem, there exists a unique $Q_{X,Y} \in \mathcal{H}_h^k$ such that $\boldsymbol{T}_{X,Y}(\varphi_h) = (\nabla Q_{X,Y}, \nabla\varphi_h)$. The correspondence $(X,Y) \to Q_{X,Y}$, gives the well-defined mapping $\mathcal{L}_1 : \mathcal{H}_h^k \times \mathcal{H}_h^k \to \mathcal{H}_h^k$ satisfying (5.2.22). Analogously we can obtain the mapping $\mathcal{L}_2$ is also well-defined.

First we prove function $\mathcal{L}$ is continuous on $\mathcal{H}_h^k \times \mathcal{H}_h^k$. Consider a vector $(X_1, Y_1) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$. For some $(X_2, Y_2) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$, denote $\boldsymbol{L} := \mathcal{L}(X_1,Y_1) - \mathcal{L}(X_2,Y_2)(:= (\Theta,\Upsilon)$ say). Then using (5.2.22) and (5.2.23), we have,

$$
\begin{aligned}
|||\boldsymbol{L}|||^2 &= [\mathcal{L}(X_1,Y_1) - \mathcal{L}(X_2,Y_2), \boldsymbol{L}] = [\mathcal{L}(X_1,Y_1), \boldsymbol{L}] - [\mathcal{L}(X_2,Y_2), \boldsymbol{L}] \\
&= \big(\nabla\mathcal{L}_1(X_1,Y_1) - \nabla\mathcal{L}_1(X_2,Y_2),\, \nabla\Theta\big) + \big(\nabla\mathcal{L}_2(X_1,Y_1) - \nabla\mathcal{L}_2(X_2,Y_2),\, \nabla\Upsilon\big) \\
&= m_h(X_1 - X_2, \Theta) + \Delta t\, \mathcal{A}_1(g_1(\Pi_k^0 X_1), g_2(\Pi_k^0 Y_1))\, a_h(X_1,\Theta) \\
&\quad -\Delta t\, \mathcal{A}_1(g_1(\Pi_k^0 X_2), g_2(\Pi_k^0 Y_2))\, a_h(X_2,\Theta) + (\Delta t)\langle f_{1h}(X_2,Y_2) - f_{1h}(X_1,Y_1), \Theta\rangle \\
&\quad +m_h(Y_1 - Y_2, \Upsilon) + \Delta t\, \mathcal{A}_2(g_1(\Pi_k^0 X_1), g_2(\Pi_k^0 Y_1))\, a_h(Y_1,\Upsilon) \\
&\quad -\Delta t\, \mathcal{A}_2(g_1(\Pi_k^0 X_2), g_2(\Pi_k^0 Y_2))\, a_h(Y_2,\Upsilon) + (\Delta t)\langle f_{2h}(X_2,Y_2) - f_{2h}(X_1,Y_1), \Upsilon\rangle
\end{aligned}
\tag{5.2.24}
$$

Using (5.2.5), Cauchy-Schwarz inequality, H$\ddot{o}$lder's inequality, Young's inequality and Poincar$\acute{e}$ inequality (with constant C_P), we get

$$
\begin{aligned}
I &:= m_h(X_1 - X_2, \Theta) + m_h(Y_1 - Y_2, \Upsilon) \\
&\leq \sum_{E\in\sum_h} \beta^*(\|X_1 - X_2\|_0\, \|\Theta\|_0 + \|Y_1 - Y_2\|_0\, \|\Upsilon\|_0) \\
&\leq \beta^* \sum_{E\in\sum_h} (\|X_1 - X_2\|_0 + \|Y_1 - Y_2\|_0)\, (\|\Theta\|_0 + \|\Upsilon\|_0) \\
&\leq \beta^* \Big(\sum_{E\in\sum_h} (\|X_1 - X_2\|_0 + \|Y_1 - Y_2\|_0)^2\Big)^{\frac{1}{2}} \Big(\sum_{E\in\sum_h} (\|\Theta\|_0 + \|\Upsilon\|_0)^2\Big)^{\frac{1}{2}} \\
&\leq 2\beta^* \Big(\sum_{E\in\sum_h} (\|X_1 - X_2\|_0^2 + \|Y_1 - Y_2\|_0^2)\Big)^{\frac{1}{2}} \Big(\sum_{E\in\sum_h} (\|\Theta\|_0^2 + \|\Upsilon\|_0^2)\Big)^{\frac{1}{2}} \\
&\leq 2\beta^*\, C_P^2 \Big(\sum_{E\in\sum_h} (\|\nabla(X_1 - X_2)\|_0^2 + \|\nabla(Y_1 - Y_2)\|_0^2)\Big)^{\frac{1}{2}} \Big(\sum_{E\in\sum_h} (\|\nabla\Theta\|_0^2 + \|\nabla\Upsilon\|_0^2)\Big)^{\frac{1}{2}} \\
&\leq 2\beta^*\, C_P^2\, |||(X_1 - X_2, Y_1 - Y_2)|||\; |||\boldsymbol{L}|||.
\end{aligned}
\tag{5.2.25}
$$

Let $II := \Delta t\, \mathcal{A}_1(g_1(\Pi_k^0 X_1), g_2(\Pi_k^0 Y_1))\, a_h(X_1,\Theta) - \Delta t\, \mathcal{A}_1(g_1(\Pi_k^0 X_2), g_2(\Pi_k^0 Y_2))\, a_h(X_2,\Theta)$.

Adding and subtracting $\Delta t\, \mathcal{A}_1(g_1(\Pi_k^0 X_2), g_2(\Pi_k^0 Y_2)) a_h(X_1, \Theta)$ to II, we get

$$
\begin{aligned}
II &= \Delta t \Big(\mathcal{A}_1(g_1(\Pi_k^0 X_1), g_2(\Pi_k^0 Y_1)) - \mathcal{A}_1(g_1(\Pi_k^0 X_2), g_2(\Pi_k^0 Y_2)) \Big)\, a_h(X_1, \Theta) \\
&\quad + \Delta t\, \mathcal{A}_1(g_1(\Pi_k^0 X_2), g_2(\Pi_k^0 Y_2))\, a_h(X_1 - X_2, \Theta).
\end{aligned}
$$

Using Assumption 5.1, (5.2.5) and Poincar$\acute{e}$ inequality, we get

$$
\begin{aligned}
II &\leq \Delta t \Big(L_A\, C_P (\|\nabla(X_1 - X_2)\|_0 + \|\nabla(Y_1 - Y_2)\|_0)\, \alpha^* \, \|\nabla X_1\|_0 \, \|\nabla \Theta\|_0 \\
&\quad + M\, \alpha^* \, \|\nabla(X_1 - X_2)\|_0 \, \|\nabla \Theta\|_0 \Big) \\
&\leq \mathcal{C}(X_1, Y_1)\, (\|\nabla(X_1 - X_2)\|_0 + \|\nabla(Y_1 - Y_2)\|_0)\, (\|\nabla \Theta\|_0 + \|\nabla \Upsilon\|_0), \quad (5.2.26)
\end{aligned}
$$

where $\mathcal{C}(X_1, Y_1) := \Delta t\, (L_A\, C_P\, \alpha^* \, |||(X_1, Y_1)||| + M\, \alpha^*)$. Similarly,

$$
\begin{aligned}
III &= \Delta t\, \mathcal{A}_2(g_1(\Pi_k^0 X_1), g_2(\Pi_k^0 Y_1))\, a_h(Y_1, \Upsilon) - \Delta t\, \mathcal{A}_2(g_1(\Pi_k^0 X_2), g_2(\Pi_k^0 Y_2))\, a_h(Y_2, \Upsilon) \\
&\leq \mathcal{C}(X_1, Y_1)\, (\|\nabla(X_1 - X_2)\|_0 + \|\nabla(Y_1 - Y_2)\|_0)\, (\|\nabla \Theta\|_0 + \|\nabla \Upsilon\|_0). \quad (5.2.27)
\end{aligned}
$$

Adding (5.2.26) and (5.2.27) we obtain

$$
\begin{aligned}
II + III &\leq 2\, \mathcal{C}(X_1, Y_1)\, (\|\nabla(X_1 - X_2)\|_0 + \|\nabla(Y_1 - Y_2)\|_0)\, (\|\nabla \Theta\|_0 + \|\nabla \Upsilon\|_0) \\
&\leq 4\, \mathcal{C}(X_1, Y_1)\; |||(X_1 - X_2, Y_1 - Y_2)|||\; |||\boldsymbol{L}|||. \quad (5.2.28)
\end{aligned}
$$

Using assumption 5.1 and Poincar$\acute{e}$ inequality, we have

$$
\begin{aligned}
&(\Delta t)\langle f_{1h}(X_2, Y_2) - f_{1h}(X_1, Y_1), \Theta \rangle + (\Delta t)\langle f_{2h}(X_2, Y_2) - f_{2h}(X_1, Y_1), \Upsilon \rangle \\
\leq\ & L_F\, C_P^2\, (\|\nabla(X_1 - X_2)\|_0 + \|\nabla(Y_1 - Y_2)\|_0)\, (\|\nabla \Theta\|_0 + \|\nabla \Upsilon\|_0) \\
\leq\ & L_F\, C_P^2\, 2\, |||(X_1 - X_2, Y_1 - Y_2)|||\; |||\boldsymbol{L}|||. \quad (5.2.29)
\end{aligned}
$$

Substituting (5.2.25), (5.2.28) and (5.2.29) into (5.2.24), we obtain

$$
|||\mathcal{L}(X_1, Y_1) - \mathcal{L}(X_2, Y_2)||| \leq \Big(2\beta^* \, C_P^2 + 4\mathcal{C}(X_1, Y_1) + 2L_F\, C_P^2 \Big)\, |||(X_1 - X_2, Y_1 - Y_2)|||. \quad (5.2.30)
$$

Hence, given $(X_1, Y_1) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ and $\epsilon > 0$, choose $\delta = \dfrac{\epsilon}{2\beta^* \, C_P^2 + 4\mathcal{C}(X_1, Y_1) + 2L_F\, C_P^2}$. Then for $(X_2, Y_2) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ whenever $|||(X_1 - X_2, Y_1 - Y_2)||| < \delta$, (5.2.30) implies $|||\boldsymbol{L}||| < \epsilon$. This proves $\mathcal{L}$ is continuous.

Consider any $(Z_1, Z_2) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ with

$$|||(Z_1, Z_2)||| = \frac{3 + 4\,C\,\beta^*\,|||(U^{n-1}, V^{n-1})|||}{C\,(\,\beta_* + \Delta t m_0 \alpha_*\,)} =: R. \tag{5.2.31}$$

Next, we derive that $[\mathcal{L}(Z_1, Z_2), (Z_1, Z_2)] > 0$. Note that

$$[\mathcal{L}(Z_1, Z_2), (Z_1, Z_2)] := (\,\nabla\mathcal{L}_1(Z_1, Z_2), \nabla Z_1\,) + (\,\nabla\mathcal{L}_2(Z_1, Z_2), \nabla Z_2\,).$$

Using (5.2.22), we obtain:

$$\begin{aligned}
(\,\nabla\mathcal{L}_1(Z_1, Z_2), \nabla Z_1\,) \;&:=\; m_h(Z_1, Z_1) + \Delta t\,\mathcal{A}_1(g_1(\Pi_k^0 Z_1), g_2(\Pi_k^0 Z_2))\,a_h(Z_1, Z_1) \\
&\qquad -\Delta t\,\langle f_{1h}(Z_1, Z_2), Z_1\rangle - m_h(U^{n-1}, Z_1) \\
&\geq\; \beta_*\|Z_1\|_0^2 + \Delta t m_0\,\alpha_*\,\|\nabla Z_1\|_0^2 - C(L_F)\Delta t\Big(\|Z_1\|_0 + \|Z_2\|_0 \\
&\qquad +|f_1(0,0)|\Big)\|Z_1\|_0 - \beta^*\,\|U^{n-1}\|_0\,\|Z_1\|_0.
\end{aligned} \tag{5.2.32}$$

Similarly, from (5.2.23) we derive

$$\begin{aligned}
(\,\nabla\mathcal{L}_2(Z_1, Z_2), \nabla Z_2\,) \;&:=\; m_h(Z_2, Z_2) + \Delta t\,\mathcal{A}_2(g_1(\Pi_0^k Z_1), g_2(\Pi_0^k Z_2))\,a_h(Z_2, Z_2) \\
&\qquad -\Delta t\,\langle f_{2h}(Z_1, Z_2), Z_2\rangle - m_h(V^{n-1}, Z_2) \\
&\geq\; \beta_*\|Z_2\|_0^2 + \Delta t\,m_0\,\alpha_*\,\|\nabla Z_2\|_0^2 - \Delta t C(L_F)\Big(\|Z_1\|_0 + \|Z_2\|_0 \\
&\qquad +|f_2(0,0)|\Big)\;\|Z_2\|_0 - \beta^*\,\|V^{n-1}\|_0\,\|Z_2\|_0.
\end{aligned} \tag{5.2.33}$$

Adding (5.2.32) and (5.2.33), and using equivalance of norms $\|\cdot\|_0$, $\|\nabla\cdot\|_0$ on $\mathcal{H}_h^k$, we have

$$\begin{aligned}
&[\mathcal{L}(Z_1, Z_2), (Z_1, Z_2)] \geq C\,(\,\beta_* + \Delta t m_0\alpha_*\,)\,(\|\nabla Z_1\|_0^2 + \|\nabla Z_2\|_0^2) \\
&\quad -\Delta t C(L_F)\Big(\,2\sqrt{2}\,(\|\nabla Z_1\|_0^2 + \|\nabla Z_2\|_0^2)^{\frac{1}{2}} + |f_1(0,0)| + |f_2(0,0)|\Big)\;(\|\nabla Z_1\|_0^2 + \|\nabla Z_2\|_0^2)^{\frac{1}{2}} \\
&\quad -2\,C\,\beta^*\;(\|\nabla U^{n-1}\|_0^2 + \|\nabla V^{n-1}\|_0^2)^{\frac{1}{2}}\;(\|\nabla Z_1\|_0^2 + \|\nabla Z_2\|_0^2)^{\frac{1}{2}} \\
\geq\;& |||(Z_1, Z_2)|||\,\Big[\,C\,(\,\beta_* + \Delta t m_0\alpha_*\,)\,|||(Z_1, Z_2)||| - \Delta t\,\sqrt{2}C(L_F)\Big(\,2\sqrt{2}\,|||(Z_1, Z_2)||| \\
&\quad +|f_1(0,0)| + |f_2(0,0)|\Big)\;-2\,C\,\beta^*\;|||(U^{n-1}, V^{n-1})|||\,\Big] \\
\geq\;& R\,\Big[\,C\,(\,\beta_* + \Delta t m_0\alpha_*\,)\,R - \Delta t\,\sqrt{2}C(L_F)\Big(\,2\sqrt{2}\,R + |f_1(0,0)| + |f_2(0,0)|\Big) \\
&\quad -2\,C\,\beta^*\;|||(U^{n-1}, V^{n-1})|||\,\Big]. \qquad\qquad (\,\text{use(5.2.31)}\,)
\end{aligned}$$

Choose (Δt) sufficiently small such that

$$\Delta t\,\sqrt{2}C(L_F)\Big(\,2\sqrt{2}\,R + |f_1(0,0)| + |f_2(0,0)|\Big) < 1.$$

Therefore,

$$[\mathcal{L}(Z_1,Z_2),(Z_1,Z_2)] \;\;\geq\;\; R\Big[\,C\,(\,\beta_* + \Delta t m_0\alpha_*\,)\,R - 1 - 2\,C\,\beta^*\,|||(U^{n-1},V^{n-1})|||\,\Big].$$

Hence, using (5.2.31) we obtain $[\mathcal{L}(Z_1,Z_2),(Z_1,Z_2)] > 0$ for all (Z_1,Z_2) with $|||(Z_1,Z_2)||| = R$ and for sufficiently small values of Δt. Then, by Brouwer's fixed point theorem, we can assure the existence of a $(U^n,V^n) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ with $\mathcal{L}(U^n,V^n) := (0,0)$. Then substituting $\mathcal{L}_1(U^n,V^n) = 0$ in (5.2.22) and $\mathcal{L}_2(U^n,V^n) = 0$ in (5.2.23) implies that (U^n,V^n) solves the system (5.2.16) - (5.2.18) at $t = t_n$.

Now, we will prove the uniqueness. Let (U_1^n,V_1^n) and $(U_2^n,V_2^n) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ be two solutions of (5.2.16)-(5.2.17) at the n^{th} time step. Then, from (5.2.16), we have

$$m_h(U_1^n - U_2^n,\varphi_h) + \Delta t\mathcal{A}_1\Big(g_1(\Pi_0^k U_1^n), g_2(\Pi_0^k V_1^n)\Big)\;a_h(U_1^n,\varphi_h)$$
$$-\,\Delta t\mathcal{A}_1\Big(g_1(\Pi_0^k U_2^n), g_2(\Pi_0^k V_2^n)\Big)\;a_h(U_2^n,\varphi_h) + \Delta t\langle f_{1h}(U_2^n,V_2^n) - f_{1h}(U_1^n,V_1^n),\;\varphi_h\rangle = 0. \quad (5.2.34)$$

In an analogous way, we derive

$$m_h(V_1^n - V_2^n,\psi_h) + \Delta t\mathcal{A}_2\Big(g_1(\Pi_k^0 U_1^n), g_2(\Pi_k^0 V_1^n)\Big)\;a_h(V_1^n,\psi_h)$$
$$-\,\Delta t\mathcal{A}_2\Big(g_1(\Pi_k^0 U_2^n), g_2(\Pi_k^0 V_2^n)\Big)\;a_h(V_2^n,\psi_h) + \Delta t\langle f_{2h}(U_2^n,V_2^n) - f_{2h}(U_1^n,V_1^n),\;\psi_h\rangle = 0. \quad (5.2.35)$$

For better readability, we introduce the following notation: $\tau := U_1^n - U_2^n$ and $\chi := V_1^n - V_2^n$. Further, we choose the test function $\varphi_h = \tau$ and inserting in equations (5.2.34), we have

$$m_h(\tau,\tau) + \Delta t\;\mathcal{A}_1\Big(g_1(\Pi_k^0 U_1^n), g_2(\Pi_k^0 V_1^n)\Big)\;a_h(U_1^n,\tau)$$
$$-\,\Delta t\mathcal{A}_1\Big(g_1(\Pi_k^0 U_2^n), g_2(\Pi_k^0 V_2^n)\Big)\;a_h(U_2^n,\tau) + \Delta t\langle f_{1h}(U_2^n,V_2^n) - f_{1h}(U_1^n,V_1^n),\;\tau\rangle = 0. \quad (5.2.36)$$

An application of Cauchy-Schwarz inequality, Lipschitz continuity of f_1 from assumption 5.1 and the boundedness of the projection operator Π_k^0, yields

$$|\langle f_{1h}(U_2^n,V_2^n)-f_{1h}(U_1^n,V_1^n),\,\tau\rangle\,|\le C\,L_F\,\Big(\|\tau\|_0+\|\chi\|_0\Big)\,\|\tau\|_0. \tag{5.2.37}$$

Adding and subtracting $\mathcal{A}_1\Big(g_1(\Pi_0^k U_1^n), g_2(\Pi_0^k V_1^n)\Big)a_h(U_2^n,\tau)$, we rewrite the difference of the nonlocal terms in the following way:

$$\begin{aligned}
&\mathcal{A}_1\Big(g_1(\Pi_k^0U_1^n),g_2(\Pi_k^0V_1^n)\Big)a_h(U_1^n,\tau)-\mathcal{A}_1\Big(g_1(\Pi_k^0U_2^n),g_2(\Pi_k^0V_2^n)\Big)a_h(U_2^n,\tau)\\
&=\underbrace{\mathcal{A}_1\Big(g_1(\Pi_k^0U_1^n),g_2(\Pi_k^0V_1^n)\Big)a_h(U_1^n-U_2^n,\tau)}_{:=T_1}\\
&\quad+\underbrace{\Big(\mathcal{A}_1\Big(g_1(\Pi_k^0U_1^n),g_2(\Pi_k^0V_1^n)\Big)-\mathcal{A}_1\Big(g_1(\Pi_k^0U_2^n),g_2(\Pi_k^0V_2^n)\Big)\Big)\,a_h(U_2^n,\tau)}_{:=T_2}.
\end{aligned} \tag{5.2.38}$$

Using assumption 5.1 on $\mathcal{A}_1(\cdot,\cdot)$, Cauchy-Schwarz inequality and the boundedness of the operator Π_k^0, we obtain

$$m_0\,\alpha_*\,\|\nabla\tau\|_0^2\le T_1 \quad \text{and} \quad |T_2|\le C\,L_A\,(\|\tau\|_0+\|\chi\|_0)\,\|\nabla U_2^n\|_0\,\|\nabla\tau\|_0. \tag{5.2.39}$$

Substituting (5.2.37) and (5.2.39) into (5.2.36), we derive the following result:

$$m_h(\tau,\tau)+(\Delta t)\,m_0\,\alpha_*\,\|\nabla\tau\|_0^2-C\,(\Delta t)\Big(\|\tau\|_0+\|\chi\|_0\Big)\,\Big(\|\tau\|_0+\|\nabla\tau\|_0\Big)\,\le 0. \tag{5.2.40}$$

Using analogous techniques as (5.2.40), we derive from (5.2.35),

$$m_h(\chi,\chi)+(\Delta t)\,m_0\,\alpha_*\,\|\nabla\chi\|_0^2-C\,(\Delta t)\,\Big(\|\tau\|_0+\|\chi\|_0\Big)\,\Big(\|\chi\|_0+\|\nabla\chi\|_0\Big)\le 0. \tag{5.2.41}$$

Note that using the inequality $ab\le\epsilon\,a^2+\dfrac{b^2}{\epsilon}$ (with $\epsilon=\dfrac{m_0\,\alpha_*}{4}$) and the inequality $(a+b)^2\le 2a^2+2b^2$, we have

$$\begin{aligned}
\Big(\|\tau\|_0+\|\chi\|_0\Big)\,\Big(\|\tau\|_0+\|\nabla\tau\|_0\Big) &\le \frac{4}{m_0\,\alpha_*}\Big(\|\tau\|_0+\|\chi\|_0\Big)^2+\frac{m_0\,\alpha_*}{4}\Big(\|\tau\|_0+\|\nabla\tau\|_0\Big)^2\\
&\le \frac{8}{m_0\,\alpha_*}\Big(\|\tau\|_0^2+\|\chi\|_0^2\Big)+\frac{m_0\,\alpha_*}{2}\Big(\|\tau\|_0^2+\|\nabla\tau\|_0^2\Big)\\
&\le C_u(\alpha_*,m_0)\,\Big(\|\tau\|_0^2+\|\chi\|_0^2\Big)+\frac{m_0\,\alpha_*}{2}\|\nabla\tau\|_0^2.
\end{aligned} \tag{5.2.42}$$

Similarly,

$$\Big(\|\tau\|_0 + \|\chi\|_0\Big)\Big(\|\chi\|_0 + \|\nabla\chi\|_0\Big) \leq C_v(\alpha_*, m_0)\Big(\|\tau\|_0^2 + \|\chi\|_0^2\Big) + \frac{m_0\,\alpha_*}{2}\|\nabla\chi\|_0^2. \quad (5.2.43)$$

Upon adding (5.2.40) and (5.2.41), using the stability of $m_h(\cdot,\cdot)$ as in (5.2.5) and (5.2.42)-(5.2.43), we yield

$$\begin{aligned}\Big(\beta_* - C_u(\alpha_*, m_0)\Delta t\Big)\|\tau\|_0^2 + \Big(\beta_* - C_v(\alpha_*, m_0)\Delta t\Big)\|\chi\|_0^2 \\ + \frac{\Delta t\; m_0\;\alpha_*}{2}(\|\nabla\tau\|_0^2 + \|\nabla\chi\|_0^2) \leq 0.\end{aligned} \quad (5.2.44)$$

Neglecting the terms $\|\nabla\tau\|_0^2$ and $\|\nabla\chi\|_0^2$ and choosing Δt sufficiently small, we derive

$$\|\tau\|_0 + \|\chi\|_0 \leq 0. \quad (5.2.45)$$

which implies $\tau = 0$ and $\chi = 0$. □

Remark 5.5. In Theorem 5.4, we have proved the well-posedness of the fully discrete scheme at time t_n based on the assumptions that the fully discrete scheme has unique solution at each previous time steps, say $t = t_1, \cdots, t_{n-1}$.

5.2.5 Implementation of the scheme

The fully discrete formulation (5.2.16)-(5.2.18) can be solved employing Newton's method. However, the presence of the nonlocal coefficient reduces the sparse structure of the Jacobian of the nonlinear system, thereby increasing the computational cost. Since our model problem contains a coupled system, the computational cost is twice. In order to avoid this difficulty, we incorporate the idea provided in [90]. The fully discrete scheme (5.2.16)-(5.2.18) can be rewritten as

$$m_h(U^n, \varphi_h) + (\Delta t)\mathcal{A}_1(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))a_h(U^n, \varphi_h) = (\Delta t)\,\langle f_{1h}(U^n, V^n), \varphi_h\rangle + m_h(U^{n-1}, \varphi_h),$$
$$m_h(V^n, \psi_h) + (\Delta t)\mathcal{A}_2(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))a_h(V^n, \psi_h) = (\Delta t)\,\langle f_{2h}(U^n, V^n), \psi_h\rangle + m_h(V^{n-1}, \psi_h).$$

We introduce two new independent variables and rewrite the equation in the following way, let $d_1 = g_1(\Pi_k^0 U^n)$ and $d_2 = g_2(\Pi_k^0 V^n)$. Then, the above equations reduce to the following non-linear system,

$$
\begin{aligned}
&m_h(U^n,\varphi_h) + (\Delta t)\,\mathcal{A}_1(d_1,d_2)\,a_h(U^n,\varphi_h) = (\Delta t)\,\langle f_{1h}(U^n,V^n),\varphi_h\rangle + m_h(U^{n-1},\varphi_h),\\
&m_h(V^n,\psi_h) + (\Delta t)\,\mathcal{A}_2(d_1,d_2)\,a_h(V^n,\psi_h) = (\Delta t)\,\langle f_{2h}(U^n,V^n),\psi_h\rangle + m_h(V^{n-1},\psi_h),\\
&d_1 = g_1(\Pi_k^0 U^n)\\
&d_2 = g_2(\Pi_k^0 V^n).
\end{aligned} \tag{5.2.46}
$$

The Jacobian of the system (5.2.46) will be of the form

$$
J = \begin{bmatrix} A_1 & 0 & C_1 & D_1 \\ 0 & B_2 & C_2 & D_2 \\ A_3 & 0 & C_3 & 0 \\ 0 & B_4 & 0 & D_4 \end{bmatrix}_{2N^{\text{dof}}+2\times 2N^{\text{dof}}+2}
$$

where, N^{dof} represents the total number of degrees of freedom of the global virtual element space $\mathcal{H}_h^k$. In what follows, we define the residual of the fully discrete system (5.2.46) as

$$
\begin{aligned}
F_{1j} \;&:=\; m_h(U^n,\psi_j) + (\Delta t)\mathcal{A}_1(d_1,d_2)a_h(U^n,\psi_j)\\
&\quad - (\Delta t)(f_{1h}(U^n,V^n),\psi_j) - m_h(U^{n-1},\psi_j) = 0, \quad 1 \le j \le N^{\text{dof}},\\
F_{2j} \;&:=\; m_h(V^n,\psi_j) + (\Delta t)\mathcal{A}_2(d_1,d_2)a_h(V^n,\psi_j)\\
&\quad - (\Delta t)(f_{2h}(U^n,V^n),\psi_j) - m_h(V^{n-1},\psi_j) = 0, \quad 1 \le j \le N^{\text{dof}},\\
F_{1N^{\text{dof}}+1} &:= g_1(\Pi_k^0 U^n) - d_1 = 0, \quad \text{and} \quad F_{2N^{\text{dof}}+1} \;:=\; g_2(\Pi_k^0 V^n) - d_2 = 0.
\end{aligned} \tag{5.2.47}
$$

Let us define,

$$
U^n = \sum_{i=1}^{N^{\text{dof}}} \alpha_i^n \psi_i, \quad \text{and} \quad V^n = \sum_{i=1}^{N^{\text{dof}}} \beta_i^n \psi_i,
$$

where $\mathcal{B} := \{\psi_1,\ldots,\psi_{N^{\text{dof}}}\}$ forms the canonical basis of the finite dimensional space $\mathcal{H}^k(E)$, and α_i^n, and β_i^n are unknowns. Further, the entries of the Jacobian matrix are given by:

$$
\begin{aligned}
(A_1)_{ij} \;&=\; \frac{\partial F_{1j}}{\partial \alpha_i^n} = m_h(\psi_i,\psi_j) + (\Delta t)\mathcal{A}_1(d_1,d_2)a_h(\psi_i,\psi_j), \quad 1 \le i,j \le N^{\text{dof}},\\
(C_1)_{1j} \;&=\; \frac{\partial F_{1j}}{\partial d_1} = (\Delta t)\frac{\partial \mathcal{A}_1(d_1,d_2)}{\partial d_1} a_h(U^n,\psi_j), \quad 1 \le j \le N^{\text{dof}},
\end{aligned}
$$

$$\begin{aligned}
(D_1)_{1j} &= \frac{\partial F_{1j}}{\partial d_2} = (\Delta t)\frac{\partial \mathcal{A}_1(d_1,d_2)}{\partial d_2} a_h(U^n,\psi_j), \quad 1 \le j \le N^{\text{dof}}, \\
(B_2)_{ij} &= \frac{\partial F_{2j}}{\partial \beta_i^n} = m_h(\psi_i,\psi_j) + (\Delta t)\mathcal{A}_2(d_1,d_2)a_h(\psi_i,\psi_j), \quad 1 \le i,j \le N^{\text{dof}}, \\
(C_2)_{1j} &= \frac{\partial F_{2j}}{\partial d_1} = (\Delta t)\frac{\partial \mathcal{A}_2(d_1,d_2)}{\partial d_1} a_h(V^n,\psi_j), \quad 1 \le j \le N^{\text{dof}}, \\
(D_2)_{1j} &= \frac{\partial F_{2j}}{\partial d_2} = (\Delta t)\frac{\partial \mathcal{A}_2(d_1,d_2)}{\partial d_2} a_h(V^n,\psi_j), \quad 1 \le j \le N^{\text{dof}}, \\
(A_3)_{1i} &= \frac{\partial F_{1N^{\text{dof}}+1}}{\partial \alpha_i^n} = \frac{\partial g_1(\Pi_k^0 U^n)}{\partial \alpha_i^n}, \quad 1 \le i \le N^{\text{dof}}, \\
(C_3)_{11} &= \frac{\partial F_{1N^{\text{dof}}+1}}{\partial d_1} = -1, \\
(B_4)_{1i} &= \frac{\partial F_{2N^{\text{dof}}+1}}{\partial \beta_i^n} = \frac{\partial g_2(\Pi_k^0 V^n)}{\partial \beta_i^n}, \quad 1 \le i \le N^{\text{dof}}, \\
(D_4)_{11} &= \frac{\partial F_{2N^{\text{dof}}+1}}{\partial d_2} = -1.
\end{aligned}$$

Theorem 5.5. *Let assumption 5.1 and assumption 5.2 holds. Also assume that* $(U^n, V^n, d_1, d_2) \in \mathcal{H}_h^k \times \mathcal{H}_h^k \times \mathbb{R} \times \mathbb{R}$ *be the solution of the system* (5.2.46)*, then* $(U^n, V^n) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ *be the solution of* (5.2.16)-(5.2.17)*. Conversely, let* $(U^n, V^n) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ *be the solution of the system of equations* (5.2.16)-(5.2.17)*, then* $(U^n, V^n, d_1, d_2) \in \mathcal{H}_h^k \times \mathcal{H}_h^k \times \mathbb{R} \times \mathbb{R}$ *be the solution of the system* (5.2.46)*.*

Proof. Proceed similar to proof of Theorem 4.1 in [91]. □

5.3 A priori error estimate for semi-discrete scheme

In this section, we establish *a priori* error estimate for the semi discrete scheme in the L^2 and H^1 norms. It is observed that the direct computation of the error $\|u(t) - u_h(t)\|_0 + \|v(t) - v_h(t)\|_0$ may not be straightforward to bound. To achieve the goal, we introduce the Ritz projection operator $\mathcal{R}_h : H^1(\Omega) \to \mathcal{H}_h^k$ that is defined as

$$a_h(\mathcal{R}_h u, \omega) = a(u, \omega) \quad \forall \omega \in H^1(\Omega). \tag{5.3.1}$$

The Ritz projection operator $\mathcal{R}_h$ directly follows from the coercivity and boundedness of the bilinear form $a_h(\cdot,\cdot)$ and the continuity of the function $a(u,\cdot)$ on $\mathcal{H}_h^k$. Employing the projection operator $\mathcal{R}_h$, we bisect the error $u(t) - u_h(t)$ and $v(t) - v_h(t)$ into two parts as

$$u(\cdot,t) - u_h(\cdot,t) = \underbrace{u(\cdot,t) - \mathcal{R}_h u(t)}_{=:\rho_1} - \underbrace{(-\mathcal{R}_h u(t) + u_h(\cdot,t))}_{=:\rho_2}, \tag{5.3.2}$$

$$v(\cdot,t) - v_h(\cdot,t) = \underbrace{v(\cdot,t) - \mathcal{R}_h v(t)}_{=:\mu_1} - \underbrace{(-\mathcal{R}_h v(t) + v_h(\cdot,t))}_{=:\mu_2}. \tag{5.3.3}$$

Using the approximation properties of $\mathcal{R}_h$, we bound the term ρ_1, μ_1. To bound the other terms ρ_2, μ_2, we use the semi-discrete formulation (5.2.6)-(5.2.7) and the approximation properties of the projection operators on the polynomial space that will be discussed in forthcoming theorems. Next, we introduce the approximation properties of the polynomial projection operator u_π (refer [94]).

Lemma 5.1. *Consider that assumption 5.2 holds on the discretized domain. Then, for all $E \in \Sigma_h$, where $0 < h \leq 1$, and $v \in H^s(E)$, where $1 \leq s \leq k+1$, there exists a polynomial $v_\pi \in \mathbb{P}_k(E)$ such that:*

$$\|v - v_\pi\|_{0,E} + h_E \|\nabla v - \nabla v_\pi\|_{0,E} \leq C\, h_E^s\, |v|_{s,E}, \tag{5.3.4}$$

where, the positive generic constant C depends on the mesh regularity parameter γ, order k of the polynomial space $\mathbb{P}_k(E)$, but is independent of the mesh size h_E.

Let I_h be a interpolation operator on the virtual element space $\mathcal{H}_h^k$. For each element $E \in \Sigma_h$, and for $v \in H^1(\Omega)$, there exists an element $I_h^E v \in \mathcal{H}^k(E)$ such that:

$$\text{dof}_i(v) = \text{dof}_i(I_h^E v) \quad 1 \leq i \leq N_E^{\text{dof}},$$

where, N_E^{dof} denotes the total numbers of DoFs in $\mathcal{H}^k(E)$. The global interpolation operator I_h is defined such that it is reduced to I_h^E when restricted to an element E,i.e. $I_h|_E = I_h^E$. The approximation properties of the global interpolation operator is now presented below (see [30]).

Lemma 5.2. *Let assumption 5.2 hold on the discretization of the computational domain Ω. Further, we assume that $v \in H^s(\Omega)$. Then, for $1 \leq s \leq k+1$, the following approximation property holds*

$$\|v - I_h^E v\|_{0,E} + h\, \|\nabla v - \nabla I_h^E v\|_{0,E} \leq\; C h^s |v|_{s,E}, \tag{5.3.5}$$

where the constant C depends on mesh regularity parameter γ but is independent of h.

Using the interpolation operator I_h, we can prove that the Ritz projection operator approximates optimally .

Lemma 5.3. *Let $u \in H^k(\Omega)$. Then, there exists an unique functions $\mathcal{R}_h u \in \mathcal{H}_h^k$ such that*

$$\|u - \mathcal{R}_h u\|_\alpha \leq \; Ch^{\beta-\alpha}|u|_\beta, \quad \alpha = 0, 1 \text{ and } \alpha \leq \beta \leq k+1. \tag{5.3.6}$$

For interested readers, we refer to [95, Lemma 3.1] for a detailed discussion. Now we prove optimal order convergence results for the semi-discrete virtual element formulation (5.2.6)-(5.2.7), with respect to L^2 norm and H^1 semi-norm.

Theorem 5.6. *Let $(u(t), v(t)) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be the solution of the system* (5.1.8)-(5.1.11) *and let $(u_h(t), v_h(t)) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ be the discrete solution of the problem* (5.2.6)-(5.2.7)*. Further, assume that $\|u\|_{L^2(0,T;H^{k+1}(\Omega))} < \infty$, $\|v\|_{L^2(0,T;H^{k+1}(\Omega))} < \infty$, $\|D_t u\|_{L^2(0,T;H^{k+1}(\Omega))} < \infty$, $\|D_t v\|_{L^2(0,T;H^{k+1}(\Omega))} < \infty$, and $\|f(u,v)\|_{L^2(0,T;H^{k+1}(\Omega))} < \infty$. Then, for almost all $t \in (0, T]$, there exists a positive constant C which depends on the mesh regularity parameter γ, the order of the virtual element space k, the stability parameter of the discrete bilinear forms $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, but independent of the mesh size h such that, we have*

$$\begin{aligned}
&\|u_h(t) - u(t)\|_0 + \|v_h(t) - v(t)\|_0 \leq C\Big(\|u_h(0) - u(0)\|_0 + \|v_h(0) - v(0)\|_0\Big) \\
&\quad + Ch^{k+1}\Big(|u(0)|_{k+1} + |v(0)|_{k+1} + \|u\|_{L^2(0,T;H^{k+1}(\Omega))} + \|v\|_{L^2(0,T;H^{k+1}(\Omega))} + \|D_t u\|_{L^2(0,T;H^{k+1}(\Omega))} \\
&\quad + \|D_t v\|_{L^2(0,T;H^{k+1}(\Omega))} + \|f_1(u,v)\|_{L^2(0,T;H^{k+1}(\Omega))} + \|f_2(u,v)\|_{L^2(0,T;H^{k+1}(\Omega))}\Big),
\end{aligned}$$

where the initial guess $u_h(0)$ and $v_h(0)$ are chosen as $u_h(0) := I_h u(0)$ and $v_h(0) := I_h v(0)$.

Proof. Using the semi discrete scheme (5.2.6)-(5.2.7) and (5.3.1), we have

$$\begin{aligned}
&m_h(\rho_2, \varphi_h) + \Delta t\, \mathcal{A}_1\Big(g_1(\Pi_k^0 u_h), g_2(\Pi_k^0 v_h)\Big)\, a_h(\rho_2, \varphi_h) = \langle f_{1h}(u_h, v_h), \varphi_h\rangle - \langle f_1(u,v), \varphi_h\rangle \\
&\quad - m_h(D_t \mathcal{R}_h u(t), \varphi_h) + (D_t u(t), \varphi_h) + \Big(\mathcal{A}_1(g_1(u), g_2(v)) - \mathcal{A}_1(g_1(\Pi_k^0 u_h), g_2(\Pi_k^0 v_h))\Big)\, a(u(t), \varphi_h).
\end{aligned} \tag{5.3.7}$$

Using the approximation property of the L^2 projection operator Π_k^0 and the assumption 5.1, we have [96, Theorem 4.2, (23)]

$$|\langle f_{1h}(u_h, v_h), \varphi_h\rangle - \langle f_1(u,v), \varphi_h\rangle| \;\; \leq \;\; C(L_F)\Big(h^{k+1}\,|u|_{k+1} + h^{k+1}\,|v|_{k+1} + h^{k+1}|f_1(u,v)|_{k+1} + \|u-u_h\|_0 + \|v-v_h\|_0\Big)\,\|\varphi_h\|_0. \tag{5.3.8}$$

Moreover, since the nonlocal function $\mathcal{A}_1(\cdot,\cdot)$ satisfies assumption 5.1, and using the approximation properties of the L^2 projection operator Π_k^0, we derive the estimation

$$|\mathcal{A}_1(g_1(u), g_2(v)) - \mathcal{A}_1(g_1(\Pi_k^0 u_h), g_2(\Pi_k^0 v_h))| \leq C(L_A)\Big(h^{k+1}\,|u|_{k+1} + h^{k+1}\,|v|_{k+1} + \|u-u_h\|_0 + \|v-v_h\|_0\Big). \tag{5.3.9}$$

Using the polynomial consistency property of the bilinear form $m_h(\cdot,\cdot)$ and approximation properties of the L^2 projection operator and the Ritz's projection operator, we derive [95]

$$|-m_h(D_t\mathcal{R}_h u(t), \varphi_h) + (D_t u(t), \varphi_h)| \leq C\,h^{k+1}\,|D_t u|_{k+1}\|\varphi_h\|_0. \tag{5.3.10}$$

Substituting $\varphi_h = \rho_2(t)$ in (5.3.7) and using the estimations (5.3.8) - (5.3.10), and the stability property of $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, we have

$$\frac{1}{2}\frac{d}{dt}\beta_*\|\rho_2(t)\|_0^2 + C\,m_0\,\alpha_*\|\nabla\rho_2(t)\|_0^2 \;\leq\; C\,\Big(h^{k+1}|u|_{k+1} + h^{k+1}|v|_{k+1} + h^{k+1}|f_1(u,v)|_{k+1} + \|u-u_h\|_0 + \|v-v_h\|_0\Big)\,\|\rho_2(t)\|_0 + C\Big(h^{k+1}\,|u|_{k+1} + h^{k+1}\,|v|_{k+1} + \|u-u_h\|_0 + \|v-v_h\|_0\Big)\,\|\Delta u(t)\|_0\,\|\rho_2(t)\|_0 + C\,h^{k+1}\,|D_t u|_{k+1}\,\|\rho_2(t)\|_0. \tag{5.3.11}$$

Further, we decompose the error $u(t)-u_h(t)$ in the rhs of (5.3.11) into $\rho_1(t)$ and $\rho_2(t)$, and $v(t)-v_h(t)$ into $\mu_1(t)$ and $\mu_2(t)$ and using Lemma 5.3, we derive

$$\frac{1}{2}\,\beta_*\,\frac{d}{dt}\|\rho_2(t)\|_0^2 + C\alpha_*\,m_0\,\|\nabla\rho_2(t)\|_0^2 \leq\; C\Big(\|\rho_2\|_0 + \|\mu_2\|_0 + h^{k+1}\,|u|_{k+1} + h^{k+1}|v|_{k+1} + h^{k+1}\,|f_1(u,v)|_{k+1} + h^{k+1}|D_t u|_{k+1}\Big)\|\rho_2(t)\|_0.$$

Using Young's inequality and integrating both sides from 0 to t, we have

$$\|\rho_2(t)\|_0^2 - \|\rho_2(0)\|_0^2 + C(\alpha_*, \beta_*, m_0) \int_0^t \|\nabla\rho_2(s)\|_0^2 \le \ C(\beta_*)\Big(\int_0^t (\|\rho_2(s)\|_0^2 + \|\mu_2(s)\|_0^2)\Big)$$
$$+C(\beta_*)h^{2k+2}\Big(\|u\|^2_{L^1(0,t;H^{k+1}(\Omega))} + \|v\|^2_{L^1(0,t;H^{k+1}(\Omega))} + \|f_1(u,v)\|^2_{L^1(0,t;H^{k+1}(\Omega))}$$
$$+\|D_t u\|^2_{L^1(0,t;H^{k+1}(\Omega))}\Big). \tag{5.3.12}$$

Using analogous arguments as (5.3.12), we obtain from (5.2.7)

$$\|\mu_2(t)\|_0^2 - \|\mu_2(0)\|_0^2 + C(\alpha_*, \beta_*, m_0) \int_0^t \|\nabla\mu_2(s)\|_0^2 \, ds \le \ C\Big(\int_0^t (\|\rho_2(s)\|_0^2 + \|\mu_2(s)\|_0^2 \Big)$$
$$+C(\beta_*)h^{2k+2} \ \Big(\|u\|^2_{L^1(0,t;H^{k+1}(\Omega))} + \|v\|^2_{L^1(0,t;H^{k+1}(\Omega))} + \|f_2(u,v)\|^2_{L^1(0,t;H^{k+1}(\Omega))}$$
$$+\|D_t v\|^2_{L^1(0,t;H^{k+1}(\Omega))}\Big). \tag{5.3.13}$$

Upon adding both the equations (5.3.12) and (5.3.13), and neglecting the terms $\int_0^t (\|\nabla\mu_2(s)\|^2 + \|\nabla\rho_2(s)\|^2) \, ds$ we get:

$$\|\mu_2(t)\|_0^2 - \|\mu_2(0)\|_0^2 + \|\rho_2(t)\|_0^2 - \|\rho_2(0)\|_0^2 \le \ C\Big(\int_0^t (\|\rho_2(s)\|_0^2 + \|\mu_2(s)\|_0^2)$$
$$+ \ h^{2k+2} \ \big(\ \|u\|^2_{L^1(0,t;H^{k+1}(\Omega))} + \|v\|^2_{L^1(0,t;H^{k+1}(\Omega))} + \|f_2(u,v)\|^2_{L^1(0,t;H^{k+1}(\Omega))}$$
$$+ \ \|f_1(u,v)\|^2_{L^1(0,t;H^{k+1}(\Omega))} + \|D_t v\|^2_{L^1(0,t;H^{k+1}(\Omega))} + \|D_t u\|^2_{L^1(0,t;H^{k+1}(\Omega))} \big).$$

An application of Grownwall's inequality yields

$$\|\mu_2(t)\|_0^2 + \|\rho_2(t)\|_0^2 \le \|\mu_2(0)\|_0^2 + \|\rho_2(0)\|_0^2 + C \ h^{2k+2}\Big(\|u(t)\|^2_{L^1(0,t;H^{k+1}(\Omega))} + \|v(t)\|^2_{L^1(0,t;H^{k+1}(\Omega))}$$
$$+ \ \|f_2(u,v)\|^2_{L^1(0,t;H^{k+1}(\Omega))} + \|D_t v\|^2_{L^1(0,t;H^{k+1}(\Omega))} + \|D_t u\|^2_{L^1(0,t;H^{k+1}(\Omega))}\Big).$$

Moreover, using the definition (5.3.2)-(5.3.3), the approximation property of the projection operator $\mathcal{R}_h$ in Lemma (5.3), we obtain:

$$\|u(t) - u_h(t)\|_0 + \|v(t) - v_h(t)\|_0 \le C\Big(\|u(0) - u_h(0)\|_0 + \|v(0) - v_h(0)\|_0\Big)$$
$$+ \ C \ h^{k+1}\Big(|u(0)|_{k+1} + |v(0)|_{k+1} + \|u\|_{L^1(0,T;H^{k+1}(\Omega))} + \|v\|_{L^1(0,T;H^{k+1}(\Omega))} + \|D_t u\|_{L^1(0,T;H^{k+1}(\Omega))}$$
$$+ \ \|D_t v\|_{L^1(0,T;H^{k+1}(\Omega))} + \|f_1(u,v)\|_{L^1(0,t;H^{k+1}(\Omega))} + \|f_2(u,v)\|_{L^1(0,t;H^{k+1}(\Omega))}\Big).$$

□

Theorem 5.7. *Let $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be the solution of the system* (5.1.8) -(5.1.11) *and let $(u_h(t), v_h(t)) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ be the discrete solution of the problem* (5.2.6)-(5.2.7)*. Then, under the assumption of Theorem 5.6 and for almost all $t \in (0,T]$, there exists a positive constant C which depends on the mesh regularity parameter γ, the order of the virtual element space k, the stability parameter of the discrete bilinear forms $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, but independent of the mesh size h such that, we have,*

$$\|\nabla u_h(t) - \nabla u(t)\|_0 + \|\nabla v_h(t) - \nabla v(t)\|_0 \leq C\Big(\|\nabla u_h(0) - \nabla u(0)\|_0 + \|\nabla v_h(0) - \nabla v(0)\|_0\Big)$$
$$+Ch^k\Big(|u(0)|_{k+1} + |v(0)|_{k+1} + \|u\|_{L^2(0,T;H^{k+1}(\Omega))} + \|v\|_{L^2(0,T;H^{k+1}(\Omega))} + \|D_t u\|_{L^2(0,T;H^{k+1}(\Omega))}$$
$$+\|D_t v\|_{L^2(0,T;H^{k+1}(\Omega))} + \|f_1(u,v)\|_{L^2(0,T;H^{k+1}(\Omega))} + \|f_2(u,v)\|_{L^2(0,T;H^{k+1}(\Omega))}\Big). \tag{5.3.14}$$

Proof. Recollecting the estimations (5.3.7) - (5.3.10), then substituting $\varphi_h = (\rho_2(t))_t$ in (5.3.7) and using the stability property of $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, we have

$$\frac{1}{2}\beta_*\|(\rho_2(t))_t\|_0^2 + \frac{1}{2}C\, m_0\, \alpha_*\, \frac{d}{dt}\|\nabla\rho_2(t)\|_0^2$$
$$\leq C\,\Big(h^{k+1}|u|_{k+1} + h^{k+1}|v|_{k+1} + h^{k+1}|f_1(u,v)|_{k+1} + \|u-u_h\|_0 + \|v-v_h\|_0\Big)\,\|(\rho_2(t))_t\|_0$$
$$+C\Big(h^{k+1}\,|u|_{k+1} + h^{k+1}\,|v|_{k+1} + \|u-u_h\|_0 + \|v-v_h\|_0\Big)\,\|\Delta u(t)\|_0\,\|(\rho_2(t))_t\|_0$$
$$+C\,h^{k+1}\,|D_t u|_{k+1}\,\|(\rho_2(t))_t\|_0.$$

Using Young's inequality appropriately yields

$$\frac{1}{4}\beta_*\|(\rho_2(t))_t\|_0^2 + \frac{1}{2}C\, m_0\, \alpha_*\, \frac{d}{dt}\|\nabla\rho_2(t)\|_0^2$$
$$\leq C\,\Big(h^{2(k+1)}|u|_{k+1}^2 + h^{2(k+1)}|v|_{k+1}^2 + h^{2(k+1)}|f_1(u,v)|_{k+1}^2 + h^{2(k+1)}\,|D_t u|_{k+1}^2$$
$$+\|u-u_h\|_0^2 + \|v-v_h\|_0^2\Big). \tag{5.3.15}$$

Analogously, estimating (5.2.7) we obtain

$$\frac{1}{4}\beta_*\|(\mu_2(t))_t\|_0^2 + \frac{1}{2}C\, m_0\, \alpha_*\, \frac{d}{dt}\|\nabla\mu_2(t)\|_0^2 \leq C\,\Big(h^{2(k+1)}|u|_{k+1}^2 + h^{2(k+1)}|v|_{k+1}^2$$
$$+h^{2(k+1)}|f_2(u,v)|_{k+1}^2 + h^{2(k+1)}\,|D_t v|_{k+1}^2 + \|u-u_h\|_0^2 + \|v-v_h\|_0^2\Big). \tag{5.3.16}$$

Adding (5.3.15) and (5.3.16), and neglecting the positive term $\frac{1}{4}\beta_*\,(\|(\rho_2(t))_t\|_0^2 + (\mu_2(t))_t\|_0^2)$

and using Theorem 5.6 (for estimating $\|u-u_h\|_0^2+\|v-v_h\|_0^2$), we obtain,

$$\frac{d}{dt}\|\nabla\rho_2(t)\|_0^2+\frac{d}{dt}\|\nabla\mu_2(t)\|_0^2\leq C\Big(\|u_h(0)-u(0)\|_0^2+\|v_h(0)-v(0)\|_0^2\Big)$$
$$+Ch^{2(k+1)}\Big(|u(0)|_{k+1}^2+|v(0)|_{k+1}^2+\|u\|_{L^2(0,T;H^{k+1}(\Omega))}^2+\|v\|_{L^2(0,T;H^{k+1}(\Omega))}^2+\|D_tu\|_{L^2(0,T;H^{k+1}(\Omega))}^2$$
$$+\|D_tv\|_{L^2(0,T;H^{k+1}(\Omega))}^2+\|f_1(u,v)\|_{L^2(0,T;H^{k+1}(\Omega))}^2+\|f_2(u,v)\|_{L^2(0,T;H^{k+1}(\Omega))}^2\Big).$$

Integrating above equation on both sides from 0 to t, we get

$$\|\nabla\rho_2(t)\|_0^2+\|\nabla\mu_2(t)\|_0^2\leq C\Big(\|\nabla\rho_2(0)\|_0^2+\|\nabla\mu_2(0)\|_0^2+\|u_h(0)-u(0)\|_0^2+\|v_h(0)-v(0)\|_0^2\Big)$$
$$+Ch^{2(k+1)}\Big(|u(0)|_{k+1}^2+|v(0)|_{k+1}^2+\|u\|_{L^2(0,T;H^{k+1}(\Omega))}^2+\|v\|_{L^2(0,T;H^{k+1}(\Omega))}^2+\|D_tu\|_{L^2(0,T;H^{k+1}(\Omega))}^2$$
$$+\|D_tv\|_{L^2(0,T;H^{k+1}(\Omega))}^2+\|f_1(u,v)\|_{L^2(0,T;H^{k+1}(\Omega))}^2+\|f_2(u,v)\|_{L^2(0,T;H^{k+1}(\Omega))}^2\Big). \tag{5.3.17}$$

Using the definition (5.3.2)-(5.3.3), Lemma (5.3) and (5.3.17) we obtain the desired estimate (5.3.14). □

5.4 Error estimation for fully discrete scheme

In this section, we prove $apriori$ error estimates showing optimal order convergence of solutions of the fully discrete scheme (5.2.16)-(5.2.17), with respect to L^2 norm and H^1 semi-norm, at each time step.

Theorem 5.8. *Let* $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$ *be the solution of Equations* (5.1.8)-(5.1.9) *and let* $(U^n,V^n)\in\mathcal{H}_h^k\times\mathcal{H}_h^k$ *be the solution of Equations* (5.2.16)-(5.2.18) *at time* $t_n\in[0,T]$. *Further, consider the initial guess for the independent variables* u,v *as* $U^0=I_h(u(0))$ *and* $V^0=I_h(v(0))$. *Then, there exists a positive constant* C *that is independent of the mesh diameter* h *and the time increment* Δt*, such that the following estimation holds*

$$\|U^n-u(t_n)\|_0+\|V^n-v(t_n)\|_0\leq C\Big(\|U^0-u(t_0)\|_0+\|V^0-v(t_0)\|_0\Big)+C\,h^{k+1}\Big(|u(0)|_{k+1}$$
$$+|v(0)|_{k+1}+\|u\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}+\|v\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}+\|D_tu\|_{L^1(0,t_n,H^{k+1}(\Omega))}$$
$$+\|D_tv\|_{L^1(0,t_n,H^{k+1}(\Omega))}+\|f_1(u,v)\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}+\|f_2(u,v)\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}\Big)$$
$$+C\,\Delta t\,\Big(\|D_{tt}u\|_{L^1(0,t_n,L^2(\Omega))}+\|D_{tt}v\|_{L^1(0,t_n,L^2(\Omega))}\Big).$$

Proof. To prove the fully discrete estimation, we employ (5.2.16), the definition of the Ritz projection operator, and the continuous weak formulation (5.1.8) and deduce that

$$
\begin{aligned}
&m_h\Big(\frac{\rho_2^n-\rho_2^{n-1}}{\Delta t},\varphi_h\Big)+\mathcal{A}_1\Big(g_1(\Pi_k^0U^n),g_2(\Pi_k^0V^n)\Big)\ a_h(\rho_2^n,\varphi_h)=\langle f_{1h}(U^n,V^n),\varphi_h\rangle\\
&-\langle f_1(u(t_n),v(t_n),\varphi_h\rangle-m_h\Big(\frac{\mathcal{R}_hu(t_n)-\mathcal{R}_hu(t_{n-1})}{\Delta t},\varphi_h\Big)+(D_tu(t_n),\varphi_h)\\
&+\Big(\mathcal{A}_1\Big(g_1(u(t_n)),g_2(v(t_n))\Big)-\mathcal{A}_1\Big(g_1(\Pi_k^0U^n),g_2(\Pi_k^0V^n)\Big)\Big)\ a(u(t_n),\varphi_h). \quad (5.4.1)
\end{aligned}
$$

An application of the approximation property of the projection operator Π_k^0 and using assumption 5.1, the error generated by the force function approximation is bounded as :

$$
\begin{aligned}
&|\langle f_{1h}(U^n,V^n),\varphi_h\rangle-\langle f_1(u(t_n),v(t_n)),\varphi_h\rangle|\leq Ch^{k+1}\ \Big(|u(t_n)|_{k+1}+|v(t_n)|_{k+1}\\
&\quad+|f_1(u(t_n),v(t_n))|_{k+1}\Big)\ \|\varphi_h\|_0+\Big(\|\rho_2^n\|_0+\|\mu_2^n\|_0\Big)\ \|\varphi_h\|_0.
\end{aligned} \quad (5.4.2)
$$

Further, using the same arguments as ([95], Theorem 3.3), we have

$$
\begin{aligned}
&\Big|m_h\Big(\frac{\mathcal{R}_hu(t_n)-\mathcal{R}_hu(t_{n-1})}{\Delta t},\varphi_h\Big)-(D_tu(t_n),\varphi_h)\Big|\\
&\quad\leq C\ (1/\Delta t)\Big(\underbrace{\|\Delta tD_tu(t_n)-u(t_n)-u(t_{n-1})\|_0}_{=:\eta_1^n}\\
&\quad+\underbrace{h^{k+1}\ |u(t_n)-u(t_{n-1})|_{k+1}}_{=:\eta_2^n}\Big)\ \|\varphi_h\|_0. \quad (5.4.3)
\end{aligned}
$$

Using Assumption 5.1 and Green's theorem, we obtain

$$
\begin{aligned}
&\Big|\Big(\mathcal{A}_1\Big(g_1(u(t_n)),g_2(v(t_n))\Big)-\mathcal{A}_1\Big(g_1(\Pi_k^0U^n),g_2(\Pi_k^0V^n)\Big)\Big)\Big|\ |a(u(t_n),\varphi_h)|\\
&\leq C\ \Big(h^{k+1}|u(t_n)|_{k+1}+h^{k+1}|v(t_n)|_{k+1}+\|\rho_2^n\|_0+\|\mu_2^n\|_0\Big)\ \|\Delta u(t_n)\|_0\ \|\varphi_h\|_0.
\end{aligned} \quad (5.4.4)
$$

Upon choosing $\varphi_h=\rho_2^n$ in the Equation (5.4.1) and using (5.4.2)-(5.4.4), we have

$$
\begin{aligned}
&m_h(\rho_2^n,\rho_2^n)+(\Delta t)\ C(m_0,\alpha_*,\beta_*)\ \|\nabla\rho_2^n\|_0^2\leq\ C(\beta_*)\ \Big(\eta_1^n+\eta_2^n\Big)\|\rho_2^n\|_0\\
&\quad+C(\beta_*)\ \Delta t\ h^{k+1}\ \Big(|u(t_n)|_{k+1}+|v(t_n)|_{k+1}+|f_1(u(t_n),v(t_n))|_{k+1}\Big)\|\rho_2^n\|_0\\
&\quad+C(\|\Delta u(t_n)\|_0,\beta_*)\ \Delta t\Big(\|\rho_2^n\|_0+\|\mu_2^n\|_0\Big)\ \|\rho_2^n\|_0+m_h(\rho_2^{n-1},\rho_2^n).
\end{aligned}
$$

Proceeding same as ([95], Theorem 3.3), we obtain

$$\|\rho_2^n\|_0 \leq C\|\rho_2^{n-1}\|_0 + C\Delta t\Big(\|\rho_2^n\|_0 + \|\mu_2^n\|_0\Big) + C\Delta t\, h^{k+1}\Big(|u(t_n)|_{k+1} + |v(t_n)|_{k+1} + |f_1(u(t_n), v(t_n))|_{k+1}\Big) + C\,\Big(\eta_1^n + \eta_2^n\Big), \tag{5.4.5}$$

Similarly, from (5.2.17), we get

$$\|\mu_2^n\|_0 \leq C\|\mu_2^{n-1}\|_0 + C\Delta t\Big(\|\rho_2^n\|_0 + \|\mu_2^n\|_0\Big) + C\Delta t\, h^{k+1}\Big(|u(t_n)|_{k+1} + |v(t_n)|_{k+1} + |f_2(u(t_n), v(t_n))|_{k+1}\Big) + C\,\Big(\xi_1^n + \xi_2^n\Big), \tag{5.4.6}$$

where $\xi_1^n := \|\Delta t D_t v(t_n) - v(t_n) + v(t_{n-1})\|_0$ and $\xi_2^n := h^{k+1}\,|v(t_n) - v(t_{n-1})|_{k+1}$. Applying the analogous arguments as [95, page 2124], we derive

$$\sum_{\nu=1}^{n} \eta_1^\nu \leq \Delta t\, \|D_{tt}u\|_{L^1(0,t_n;L^2(\Omega))} \quad \text{and} \quad \sum_{\nu=1}^{n} \eta_2^\nu \leq h^{k+1}\, \|D_t u\|_{L^1(0,t_n;H^{k+1}(\Omega))}. \tag{5.4.7}$$

$$\sum_{\nu=1}^{n} \xi_1^\nu \leq \Delta t\, \|D_{tt}v\|_{L^1(0,t_n;L^2(\Omega))} \quad \text{and} \quad \sum_{\nu=1}^{n} \xi_2^\nu \leq h^{k+1}\, \|D_t v\|_{L^1(0,t_n;H^{k+1}(\Omega))}. \tag{5.4.8}$$

Adding the estimates (5.4.5) and (5.4.6) and proceeding as in [96, Theorem 4.4], we get

$$\|\rho_2^n\|_0 + \|\mu_2^n\|_0 \leq C\,(1 + C\,\Delta t)^n\,\Big(\|\rho_2^0\|_0 + \|\mu_2^0\|_0\Big) + C\Big(\Delta t\, h^{k+1}\,\sum_{\nu=1}^{n}(1 + C\,\Delta t)^{n-\nu} (|f_1(u(t_\nu), v(t_\nu))|_{k+1} + |f_2(u(t_\nu), v(t_\nu))|_{k+1} + |u(t_\nu)|_{k+1} + |v(t_\nu)|_{k+1})\Big) + \sum_{\nu=1}^{n}\,(1 + C\,\Delta t)^{n-\nu}\,(\eta_1^\nu + \eta_2^\nu + \xi_1^\nu + \xi_2^\nu). \tag{5.4.9}$$

Using Taylor's series expansion of $(1 + C\Delta t)^{n-\nu}$ and noting $n\Delta t \leq N_T\Delta t \leq T$, along with the estimations (5.4.7), and (5.4.8), we derive

$$\|\rho_2^n\|_0 + \|\mu_2^n\|_0 \leq C(\|\rho_2^0\|_0 + \|\mu_2^0\|_0) + C\,h^{k+1}\Big(\|f_1(u,v)\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} + \|u\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} + \|v\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} + \|D_t u\|_{L^1(0,t_n;H^{k+1}(\Omega))} + \|D_t v\|_{L^1(0,t_n;H^{k+1}(\Omega))} + \|f_2(u,v)\|_{L^\infty(0,t_n;H^{k+1}(\Omega))}\Big) + C\,\Delta t\,\Big(\|D_{tt}u\|_{L^1(0,t_n;L^2(\Omega))} + \|D_{tt}v\|_{L^1(0,t_n;L^2(\Omega))}\Big).$$

Using the estimations of $\|\rho_1^n\|_0$ and $\|\mu_1^n\|_0$ from Lemma 5.3 and the above inequality, we obtain the desired result. □

Theorem 5.9. *Let $(u,v) \in H_0^1(\Omega)\times H_0^1(\Omega)$ be the solution of the weak formulation* (5.1.8)-(5.1.9) *and $(U^n, V^n) \in \mathcal{H}_h^k(\Omega) \times \mathcal{H}_h^k(\Omega)$ be the solution of the discrete scheme* (5.2.16)-(5.2.18). *Then, the following error estimations holds*

$$\begin{aligned}
&\|\nabla(U^n - u(t_n))\|_0 + \|\nabla(V^n - v(t_n))\|_0 \leq C\, \Big(\|\nabla U^0 - \nabla u(t_0)\|_0 + \|\nabla V^0 - \nabla v(t_0)\|_0\Big) \\
&\quad + C\, h^{k+1} \Big(|u(0)|_{k+1} + |v(0)|_{k+1} + \|u\|_{L^\infty(0,t_n,H^{k+1}(\Omega))} + \|v\|_{L^\infty(0,t_n,H^{k+1}(\Omega))} \\
&\quad + \|f_1(u,v)\|_{L^\infty(0,t_n,H^{k+1}(\Omega))} + \|f_2(u,v)\|_{L^\infty(0,t_n,H^{k+1}(\Omega))} + \|D_t u\|_{L^2(0,t_n,H^{k+1}(\Omega))} \\
&\quad + \|D_t v\|_{L^2(0,t_n,H^{k+1}(\Omega))}\Big) + C\Delta t\Big(\|D_{tt}u\|_{L^2(0,t_n;L^2(\Omega))} + \|D_{tt}v\|_{L^2(0,t_n;L^2(\Omega))}\Big).
\end{aligned}$$

Proof. The error estimation for the fully discrete solution (U^n, V^n) in the energy norm can be done by employing the Ritz projection operator.

We first decompose the term $U^n - u(t_n)$ as $U^n - u(t_n) := U^n - \mathcal{R}_h u(t_n) + \mathcal{R}_h u(t_n) - u(t_n)$. Since, we can bound the term using the approximation property of the Ritz projection operator, we will focus to estimate the term $\|\nabla U^n - \nabla \mathcal{R}_h u(t_n)\|_0$.

Using the fully discrete scheme (5.2.16) -(5.2.17), and the definition of the Ritz projection operator, we write an equation in terms of ρ_2^n as

$$\begin{aligned}
&m_h\Big(\frac{\rho_2^n - \rho_2^{n-1}}{\Delta t}, \varphi_h\Big) + \mathcal{A}_1(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\, a_h(\rho_2^n, \varphi_h) = \langle f_{1h}(U^n, V^n), \varphi_h\rangle \\
&\quad -\langle f_1(u(t_n), v(t_n)), \varphi_h\rangle - m_h(\partial_n \mathcal{R}_h u(t_n), \varphi_h) + (u_t(t_n), \varphi_h) \\
&\quad +\Big(\mathcal{A}_1(g_1(u(t_n)), g_2(v(t_n))) - \mathcal{A}_1(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\Big)\, a(u(t_n), \varphi_h). \quad (5.4.10)
\end{aligned}$$

Following (5.4.2), we can bound the term $|\langle f_{1h}(U^n, V^n), \varphi_h\rangle - \langle f_1(u(t_n), v(t_n)), \varphi_h\rangle|$. The last term of (5.4.10) can be bounded as follows

$$\begin{aligned}
&\Big|\mathcal{A}_1\big(g_1(u(t_n)), g_2(v(t_n))\big) - \mathcal{A}_1\big(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n)\big)\Big|\, |a(u(t_n), \varphi_h)| \\
&\leq C\, \Big(h^{k+1}|u(t_n)|_{k+1} + h^{k+1}|v(t_n)|_{k+1} + \|\rho_2^n\|_0 + \|\mu_2^n\|_0\Big)\, \|\Delta u(t_n)\|_0\, \|\varphi_h\|_0,
\end{aligned} \quad (5.4.11)$$

where, we have exploited assumption 5.1 and the approximation property of the operator Π_k^0. Further, following the technique mentioned in [95, (34),Theorem 3.3], we obtain

$$|-m_h(\partial_n \mathcal{R}_h u(t_n), \varphi_h) + (D_t u(t_n), \varphi_h)| \leq C\, \frac{1}{\Delta t}\Big(\eta_1^n + \eta_2^n\Big)\, \|\varphi_h\|_0. \quad (5.4.12)$$

Upon substituting $\varphi_h = \partial\rho_2^n$ in (5.4.10), and using (5.4.12), (5.4.10), and boundedness of load term, we obtain

$$m_h(\partial\rho_2^n, \partial\rho_2^n) + \frac{m_0}{2}\,\alpha_*\,\partial\|\nabla\rho_2^n\|_0^2 \leq \; C\, h^{k+1}\Big(|u(t_n)|_{k+1} + |v(t_n)|_{k+1}$$
$$+\, |f_1(u(t_n), v(t_n))|_{k+1}\Big)\, \|\partial\rho_2^n\|_0 + C\frac{1}{\Delta t}\Big(\|\eta_1^n\|_0 + \|\eta_2^n\|_0\Big)\, \|\partial\rho_2^n\|_0 + C\Big(\|\rho_2^n\|_0 + \|\mu_2^n\|_0\Big)\, \|\partial\rho_2^n\|_0.$$

Using Young's inequality, kick back arguments, and proceeding analogous arguments as in [95], we can deduce that

$$\begin{aligned} \|\nabla\rho_2^n\|_0^2 \;\; &\leq \;\; \|\nabla\rho_2^{n-1}\|_0^2 + C\, h^{2k+2}(\Delta t)\Big(|u(t_n)|_{k+1}^2 + |v(t_n)|_{k+1}^2 + |f_1(u(t_n), v(t_n))|_{k+1}^2\Big) \\ &\quad + C\frac{1}{\Delta t}(\|\eta_1^n\|_0^2 + \|\eta_2^n\|_0^2) + \Delta t\Big(\|\rho_2^n\|_0^2 + \|\mu_2^n\|_0^2\Big). \end{aligned} \tag{5.4.13}$$

Similarly from (5.2.17) and proceeding same as (5.4.13), we obtain

$$\begin{aligned} \|\nabla\mu_2^n\|_0^2 \;\; &\leq \;\; \|\nabla\mu_2^{n-1}\|_0^2 + C\, h^{2k+2}(\Delta t)\Big(|u(t_n)|_{k+1}^2 + |v(t_n)|_{k+1}^2 + |f_2(u(t_n), v(t_n))|_{k+1}^2\Big) \\ &\quad + C\frac{1}{\Delta t}(\|\xi_1^n\|_0^2 + \|\xi_2^n\|_0^2) + \Delta t\Big(\|\rho_2^n\|_0^2 + \|\mu_2^n\|_0^2\Big). \end{aligned} \tag{5.4.14}$$

Upon summing Equations (5.4.13) and (5.4.14) and letting the sum $\nu = 1, \cdots, n$, and using the estimation of $\sum_{\nu=1}^{n}\Big(\|\rho_2^\nu\|_0 + \|\mu_2^\nu\|_0\Big)$ from Theorem 5.8, we obtain the desired result. □

5.5 Error estimation for linearized scheme

In this section, we estimate the rate of convergence in the space variable as well as the time variable for the approximation $(\widetilde{U}^n, \widetilde{V}^n)$ satisfying (5.2.19)-(5.2.20). Employing the Ritz projection operator $\mathcal{R}_h$ (see (5.3.1)), we split the terms $u(t_n) - \widetilde{U}^n$ and $v(t_n) - \widetilde{V}^n$ as follows

$$u(t_n) - \widetilde{U}^n := \rho_1^n + \widetilde{\rho_2}^n; \quad v(t_n) - \widetilde{V}^n := \mu_1^n + \widetilde{\mu_2}^n.$$

Theorem 5.10. *Let* $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *be the solution of equations* (5.1.8)-(5.1.11) *and* $\{(\widetilde{U}^n, \widetilde{V}^n)\}_n \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ *be the sequence of solutions of* (5.2.16)-(5.2.18) *for time steps* $t_n \in [0, T]$. *Further, assume that the exact solution* (u, v), *and the force function* $f_i(u, v)$, $i \in \{1, 2\}$ *satisfy the regularity assumptions, i.e.* $\|u\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} < \infty$, $\|v\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} < \infty$, $\|D_t u\|_{L^1(0,t_n;H^{k+1}(\Omega))} < \infty$, $\|D_t v\|_{L^1(0,t_n;H^{k+1}(\Omega))} < \infty$, $\|D_{tt} u\|_{L^1(0,t_n;H^{k+1}(\Omega))} < \infty$, $\|D_{tt} v\|_{L^1(0,t_n;H^{k+1}(\Omega))} < \infty$, $\|f_i(u, v)\|_{L^1(0,t_n;H^{k+1}(\Omega))} < \infty$.

Then the following error estimation holds

$$\|\widetilde{U}^n - u(t_n)\|_0 + \|\widetilde{V}^n - v(t_n)\|_0 \leq C\Big(\|U^0 - u(0)\|_0 + \|V^0 - v(0)\|_0\Big) + C\ h^{k+1}\Big(\|u\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} + \|v\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} + \|D_t u\|_{L^1(0,t_n;H^{k+1}(\Omega))} + \|D_t v\|_{L^1(0,t_n;H^{k+1}(\Omega))} + \|f_1(u,v)\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} + \|f_2(u,v)\|_{L^\infty(0,t_n;H^{k+1}(\Omega))}\Big) + C\ \Delta t\Big(\|D_{tt}u\|_{L^1(0,t_n;H^{k+1}(\Omega))} + \|D_t u\|_{L^1(0,t_n;H^{k+1}(\Omega))} + \|D_{tt}v\|_{L^1(0,t_n;H^{k+1}(\Omega))} + \|D_t v\|_{L^1(0,t_n;H^{k+1}(\Omega))}\Big).$$

The positive generic constant C depends on mesh regularity γ, stability parameters of the discrete bilinear forms $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$ but is independent of the mesh parameter h and time step Δt.

Proof. Using the projection operator $\mathcal{R}_h$, we split the error as $u(t_n) - \widetilde{U}^n := \rho_1^n + \widetilde{\rho}_2^n$ and $v(t_n) - \widetilde{V}^n := \mu_1^n + \widetilde{\mu}_2^n$. The estimations of ρ_1^n and μ_1^n are known from the approximation property of $\mathcal{R}_h$. In order to estimate $\widetilde{\rho}_2^n$ and $\widetilde{\mu}_2^n$, we proceed as follows. By considering (5.2.19), we obtain

$$m_h\Big(\frac{\widetilde{\rho}_2^n - \widetilde{\rho}_2^{n-1}}{\Delta t}, \varphi_h\Big) + \mathcal{A}_1\Big(g_1(\Pi_k^0\widetilde{U}^{n-1}), g_2(\Pi_k^0\widetilde{V}^{n-1})\Big)\ a_h(\widetilde{\rho}_2^n, \varphi_h) = \langle f_{1h}(\widetilde{U}^{n-1}, \widetilde{V}^{n-1}), \varphi_h\rangle - \langle f_1(u(t_n), v(t_n)), \varphi_h\rangle - m_h\Big(\frac{\mathcal{R}_h u(t_n) - \mathcal{R}_h u(t_{n-1})}{\Delta t}, \varphi_h\Big) + (D_t u(t_n), \varphi_h) + \Big[\mathcal{A}_1\big(g_1(u(t_n)), g_2(v(t_n))\big) - \mathcal{A}_1\big(g_1(\Pi_k^0\widetilde{U}^{n-1}), g_2(\Pi_k^0\widetilde{V}^{n-1})\big)\Big]\ a(u(t_n), \varphi_h). \tag{5.5.1}$$

The load term in the right hand side can be rewritten as follows

$$\begin{aligned}
&|\langle f_{1h}(\widetilde{U}^{n-1}, \widetilde{V}^{n-1}), \varphi_h\rangle - \langle f_1(u(t_n), v(t_n)), \varphi_h\rangle| \\
&\leq |\langle f_1(\Pi_k^0\widetilde{U}^{n-1}, \Pi_k^0\widetilde{V}^{n-1}), \Pi_k^0\varphi_h\rangle - \langle f_1(\Pi_k^0 u(t_n), \Pi_k^0 v(t_n)), \Pi_k^0\varphi_h\rangle| \\
&+ |\langle f_1(\Pi_k^0 u(t_n), \Pi_k^0 v(t_n)), \Pi_k^0\varphi_h\rangle - \langle f_1(u(t_n), v(t_n)), \Pi_k^0\varphi_h\rangle| \\
&\quad + |\langle f_1(u(t_n), v(t_n)), \Pi_k^0\varphi_h\rangle - \langle f_1(u(t_n), v(t_n)), \varphi_h\rangle|.
\end{aligned} \tag{5.5.2}$$

Using assumption 5.1, the approximation property of the L^2 projection operator Π_k^0, we have:

$$\begin{aligned}
&|\langle f_{1h}(\widetilde{U}^{n-1}, \widetilde{V}^{n-1}), \varphi_h\rangle - \langle f_1(u(t_n), v(t_n)), \varphi_h\rangle| \\
&\qquad \leq\ C\Big(\|\widetilde{U}^{n-1} - u(t_n)\|_0 + \|\widetilde{V}^{n-1} - v(t_n)\|_0\Big)\ \|\varphi_h\|_0 \\
&\qquad\quad + h^{k+1}\ \Big(|u(t_n)|_{k+1} + |v(t_n)|_{k+1} + |f_1(u(t_n), v(t_n))|_{k+1}\Big)\ \|\varphi_h\|_0.
\end{aligned} \tag{5.5.3}$$

Using the analogous technique as [95, Theorem 3.3], we split the term as

$$|-m_h(\frac{\mathcal{R}_h u(t_n)-\mathcal{R}_h u(t_{n-1})}{\Delta t},\varphi_h)+(D_t u(t_n),\varphi_h)| \leq C\ \frac{1}{\Delta t}\Big(\eta_1^n+\eta_2^n\Big)\|\varphi_h\|_0. \quad (5.5.4)$$

Furthermore, the nonlocal coefficients can be decomposed as follows

$$\begin{aligned}|-\mathcal{A}_1(g_1(\Pi^0_{k-1}\widetilde{U}^{n-1}),g_2(\Pi^0_{k-1}\widetilde{V}^{n-1}))+\mathcal{A}_1(g_1(u(t_n)),g_2(v(t_n)))| \leq\ C\ \Big(\|\widetilde{\rho}_2^{n-1}\|_0\\ +h^{k+1}|u(t_n)|_{k+1}+\|D_t u\|_{L^1(t_{n-1},t_n,L^2(\Omega))}+\|\widetilde{\mu}_2^{n-1}\|_0+h^{k+1}|v(t_n)|_{k+1}+\|D_t v\|_{L^1(t_{n-1},t_n,L^2(\Omega))}\Big).\end{aligned} \quad (5.5.5)$$

Upon substituting (5.5.3) -(5.5.5) into (5.5.1), we have

$$\begin{aligned}\|\widetilde{\rho}_2^n\|_0 &\leq \|\widetilde{\rho}_2^{n-1}\|_0+C\ \Delta t\ h^{k+1}\ \Big(|u(t_n)|_{k+1}+|v(t_n)|_{k+1}+|f_1(u(t_n),v(t_n))|_{k+1}\Big)\\ &\quad +\Delta t\ \Big(\|\widetilde{U}^{n-1}-u(t_n)\|_0+\|\widetilde{V}^{n-1}-v(t_n)\|_0\Big)+C\ \Big(\eta_1^n+\eta_2^n\Big).\end{aligned} \quad (5.5.6)$$

Using same technique as above, we obtain

$$\begin{aligned}\|\widetilde{\mu}_2^n\|_0 &\leq \|\widetilde{\mu}_2^{n-1}\|_0+C\ \Delta t\ h^{k+1}\ \Big(|u(t_n)|_{k+1}+|v(t_n)|_{k+1}+|f_2(u(t_n),v(t_n))|_{k+1}\Big)\\ &\quad +\Delta t\ \Big(\|\widetilde{U}^{n-1}-u(t_n)\|_0+\|\widetilde{V}^{n-1}-v(t_n)\|_0\Big)+C\ \Big(\xi_1^n+\xi_2^n\Big).\end{aligned} \quad (5.5.7)$$

We decompose the term

$$\|\widetilde{U}^{n-1}-u(t_n)\|_0 \leq\ C\ \|\widetilde{U}^{n-1}-\mathcal{R}_h u(t_{n-1})\|_0+h^{k+1}\ |u(t_{n-1})|_{k+1}+\|u(t_{n-1})-u(t_n))\|_0. \quad (5.5.8)$$

and

$$\|\widetilde{V}^{n-1}-v(t_n)\|_0 \leq\ C\ \|\widetilde{V}^{n-1}-\mathcal{R}_h v(t_{n-1})\|_0+h^{k+1}\ |v(t_{n-1})|_{k+1}+\|v(t_{n-1})-v(t_n))\|_0. \quad (5.5.9)$$

Using the estimations (5.5.8), and (5.5.9), and adding (5.5.6) and (5.5.7), we deduce

$$\begin{aligned}\|\widetilde{\rho}_2^n\|_0+\|\widetilde{\mu}_2^n\|_0 \leq\ C\ \Big(\|U^0-u(t_0)\|_0+\|V^0-v(t_0)\|_0\Big)+C\ h^{k+1}\ \Big(\|u\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}\\ +\|v\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}+\|f_1(u,v)\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}+\|f_2(u,v)\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}+\|D_t u\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}\\ +\|D_t v\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}\Big)+C\ \Delta t\Big(\|D_{tt}u\|_{L^1(0,t_n;L^2(\Omega))}+\|D_{tt}v\|_{L^1(0,t_n;L^2(\Omega))}\Big).\end{aligned} \quad (5.5.10)$$

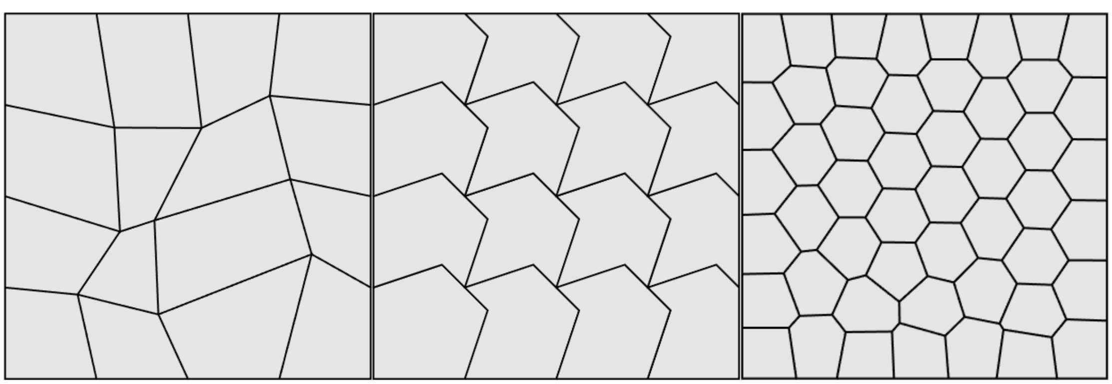
Together with (5.5.10) and an application of the estimations $\|\rho_1^n\|_0$ and $\|\mu_1^n\|_0$ (using Lemma 5.3), we obtain the desired result. $\square$

5.6 Numerical Experiments

In this section, we study the convergence and the accuracy of the virtual element method by solving a nonlocal parabolic problem for a manufactured solution. We consider a square domain, $\Omega = [0,1] \times [0,1]$. The computational domain is discretized with different type of elements, viz., distorted square, non-convex mesh and smoothed Voronoi. A few representative meshes are shown in Figure 5.1. In this study, for spatial discretization, we have considered the virtual element space of orders, $k =$ 1,2 and 3. For temporal discretization, we have employed the backward Euler time integration scheme. For convergence study, the errors are computed at the final time T in the L^2 and the H^1 norms. Since the discrete solutions are implicitly defined on the virtual space, the errors are computed using the two projection operators as follows:

$$L^2\text{-norm error: } \mathcal{E}_{h,0} := \sqrt{\sum_{E \in \Sigma_h} \|u(T) - \Pi^0_{k,E} U^{N_T}\|^2_{0,E}}.$$

$$H^1\text{-norm error: } \mathcal{E}_{h,1} := \sqrt{\sum_{E \in \Sigma_h} \|\nabla u(T) - \nabla \Pi^{\nabla}_{k,E} U^{N_T}\|^2_{0,E}}.$$



(a) Distorted Square **(b)** Non-convex **(c)** Smoothed Voronoi

Figure 5.1: A schematic representation of different discretizations employed in this study.

Consider the model problem (5.1.1)-(5.1.5), where the nonlocal coefficients are defined as:

$$\mathcal{A}_1\Big(g_1(u), g_2(v)\Big) := 3 + \cos(g_1(u)) + \sin(g_2(v))$$
$$\mathcal{A}_2\Big(g_1(u), g_2(v)\Big) := 5 - \cos(g_1(u)) + \sin(g_2(u)).$$

The force functions (f_1, f_2) are computed by imposing the following manufactured solutions:

$$u = (x - x^2)\,(y - y^2)\,e^{-t}$$
$$v = 2\,(x - x^2)\,(y - y^2)\,e^{2t}$$

as the exact solutions of (5.1.1)-(5.1.2) and $g_1(u) = \int_\Omega u\,d\Omega$, $g_2(v) = \int_\Omega v\,d\Omega$. To reduce the computational cost, one additional variable is augmented to the nonlinear system and the resulting nonlinear system is solved using the Newton's method with a user specified tolerance as $\mathcal{O}(10^{-10})$. This ensures that the sparsity of the Jacobian is retained. The nonlinear loop takes between two to five iterations for the convergence of the numerical solution. The convergence of the error in the L^2 and H^1 norms for the independent variables, u and v are shown in Figures 5.2-5.3 for k = 1,2 and k = 3, respectively. It is seen that the numerical scheme converges at an optimal order in the respective norms. In Figure 5.5, the convergence behaviour of the numerical solution obtained from the linearized scheme (5.2.19) -(5.2.20) for the virtual element space of orders k = 1,2 is shown. It is observed that the numerical solution converges optimally to the analytical solution as predicted in Theorem 5.10.

Now, we study the convergence behavior in the temporal variable t. This is done by setting the mesh parameter h = 1/80 for all the considered discretization types. The time increment is chosen as $\Delta t = 1/4, 1/8, 1/16, 1/32$. The errors are computed at the end of the each time step t_n for $n = 1, \ldots, N_T$ and added to obtain the cumulative errors up to the final time T and is given by:

$$e_{0,T,h,0} := \left(\Delta t \sum_{n=1}^{N_T}\left(\sum_{E\in\Sigma_h} \|u(t_n) - \Pi^0_{k,E} U^n\|^2_{0,E}\right)\right)^{1/2}. \tag{5.6.1}$$

In this case, we only report the results for the lowest order virtual element space, i.e., k = 1. Figure 5.4 shows the convergence of the error in the L^2 norm for both the independent variables. It can be inferred that the numerical scheme yields optimal convergence rate as predicted in Theorem 5.8. Further, it is noted that for higher order virtual element space, the numerical scheme converges at an optimal rate.

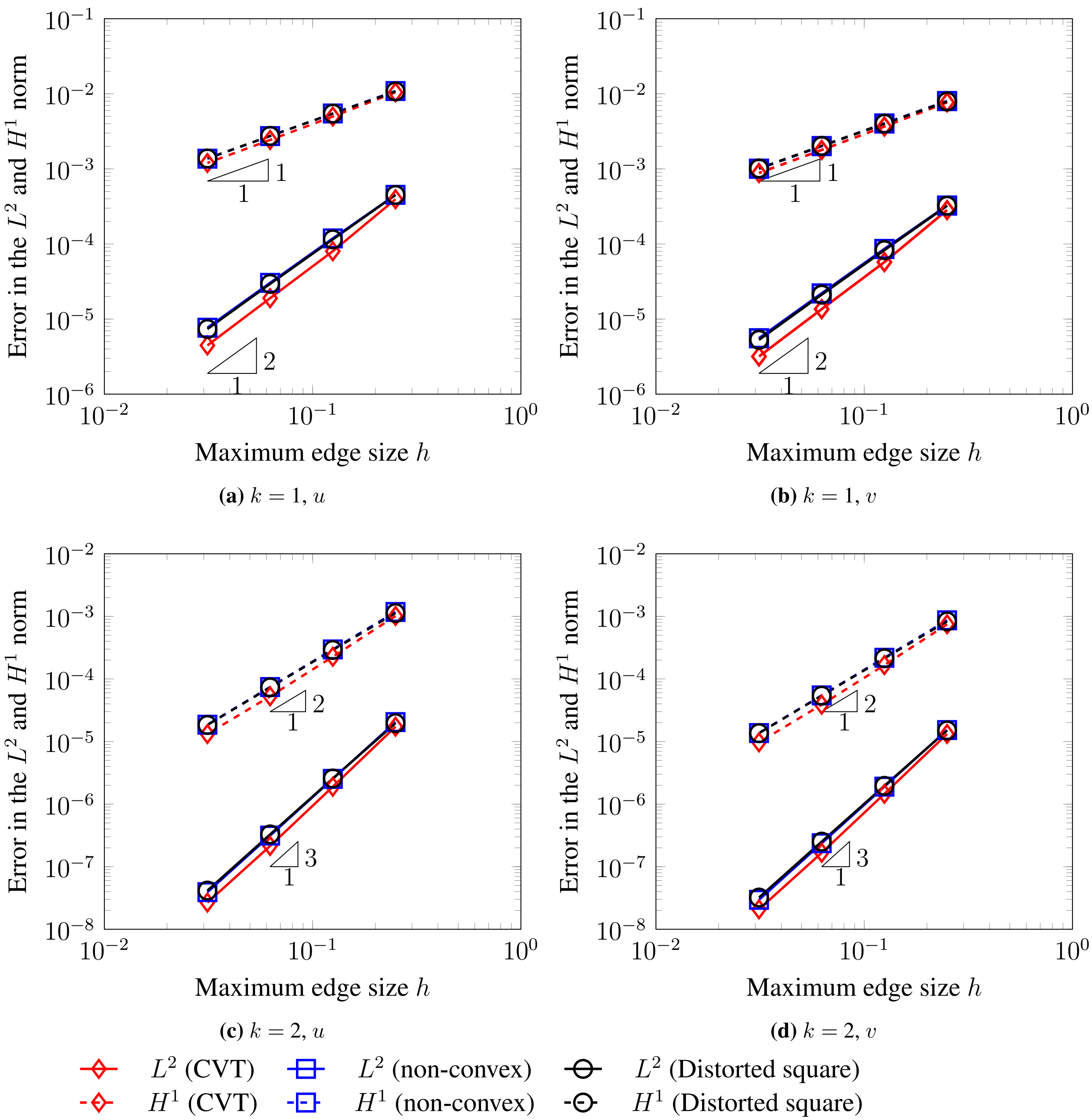
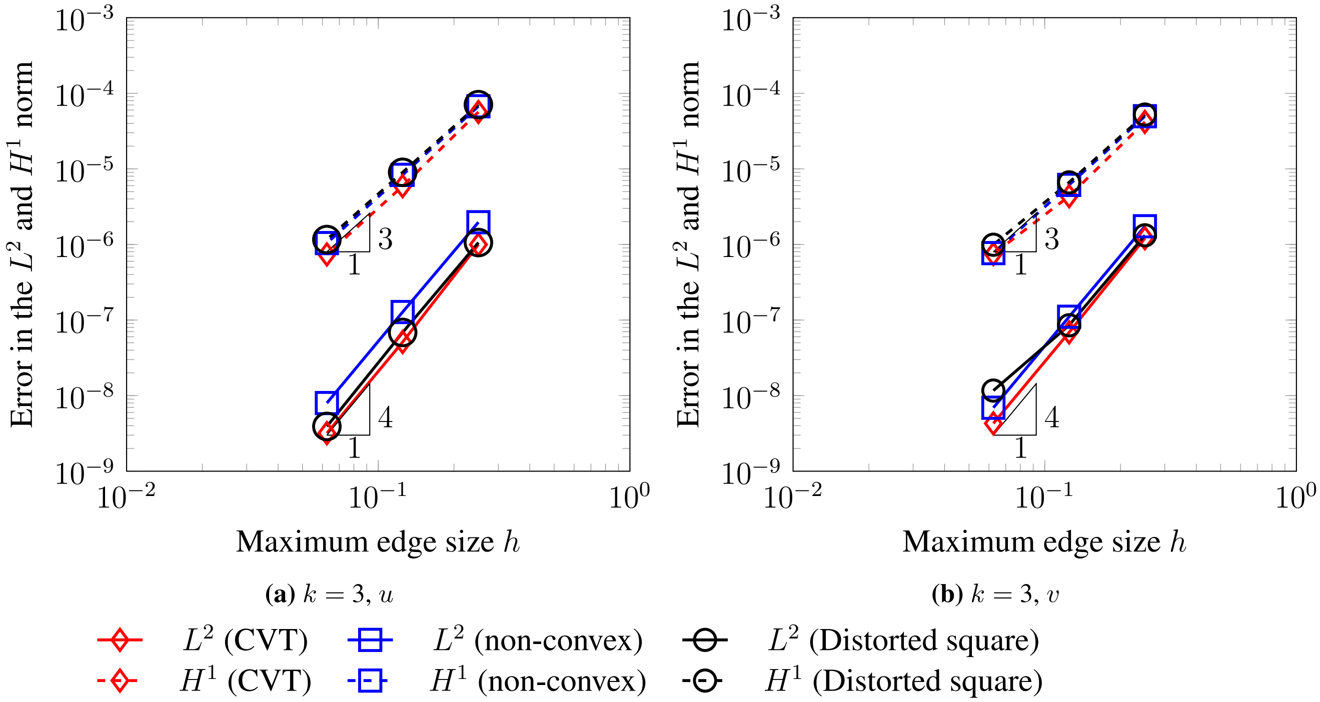
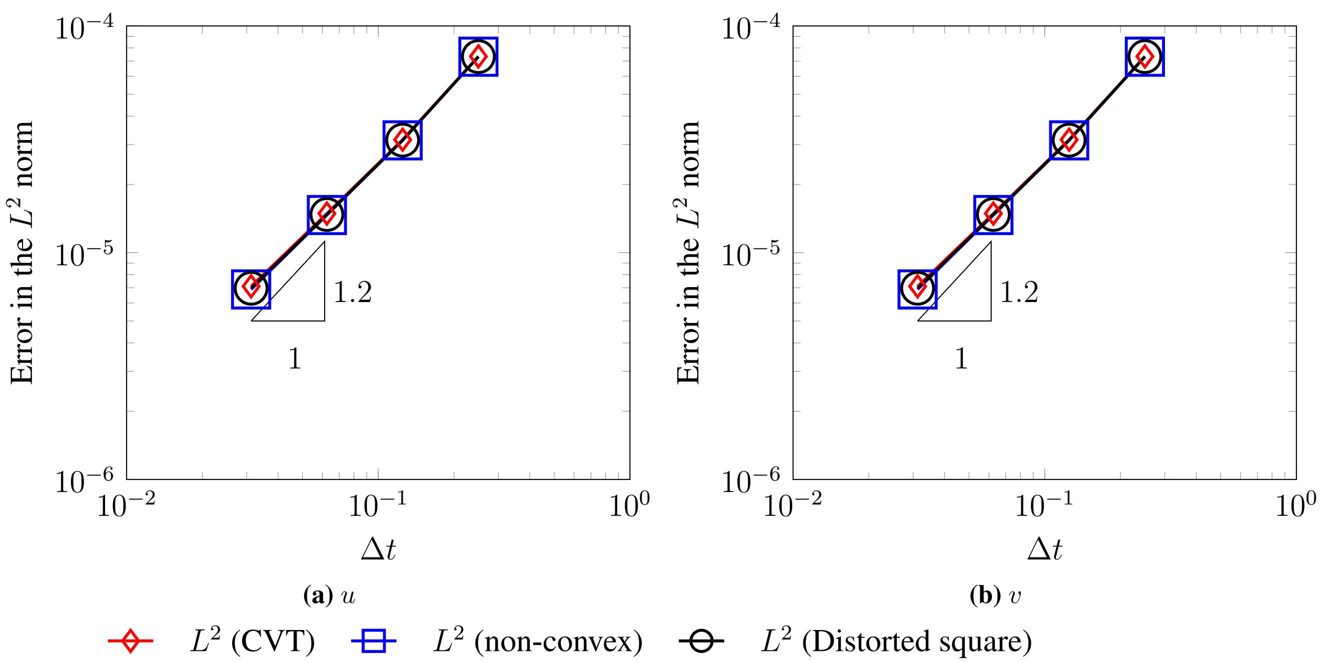


Figure 5.2: Convergence of the errors in the L^2 norm and H^1 norm for $k = 1$ and 2 and for the variables, u and v



(a) $k = 3$, u **(b)** $k = 3$, v

Figure 5.3: Convergence of the errors in the L^2 norm and H^1 norm for $k = 3$ and for the variables, u and v



(a) u **(b)** v

Figure 5.4: Convergence of the error in the L^2 norm $k = 1$ and $h = 1/80$ and for the variables, u and v

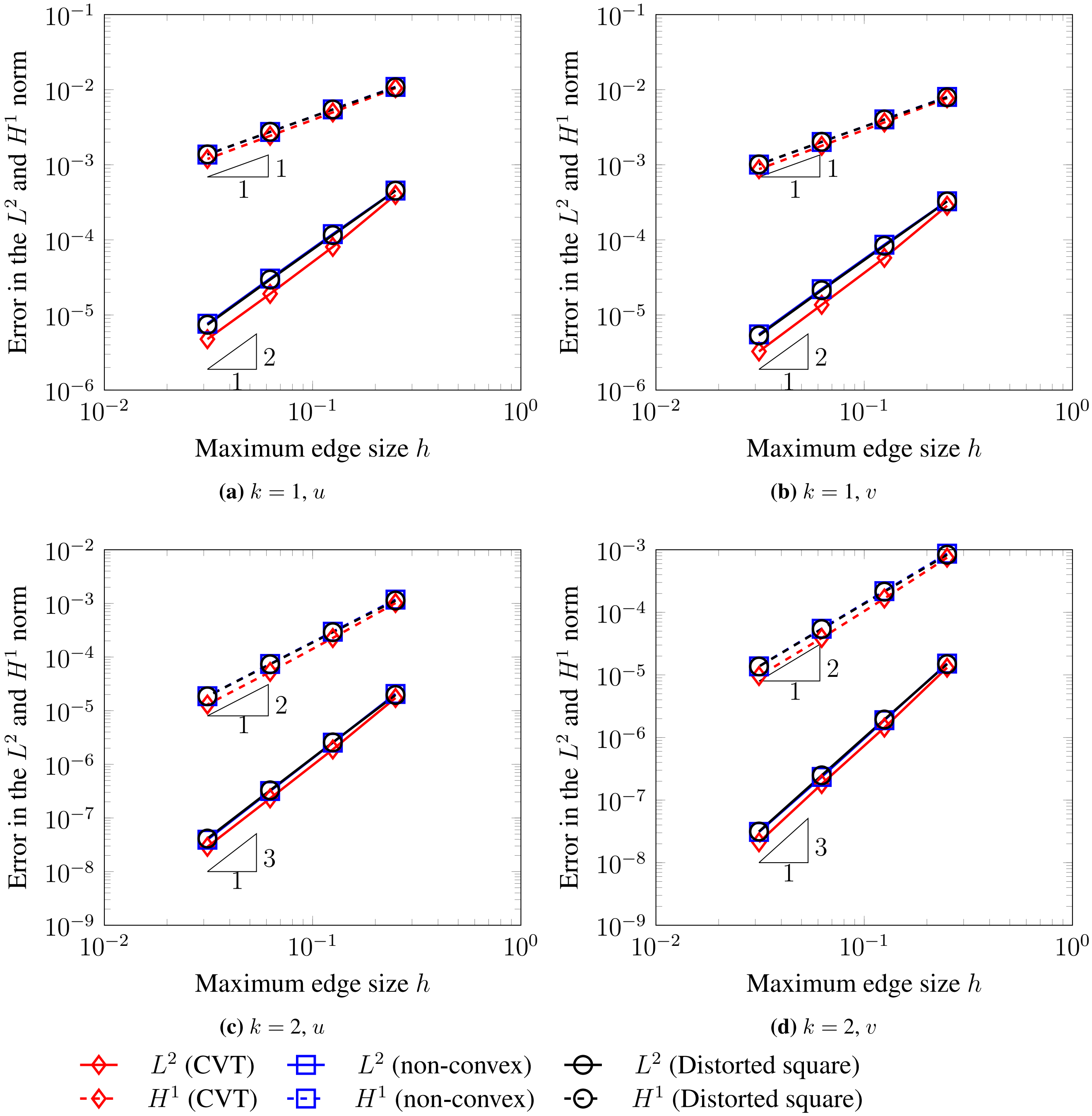


Figure 5.5: Convergence of the errors in the L^2 norm and H^1 norm for $k = 1$ and 2 and for the variables, u and v for the linearized scheme

5.7 Summary

In this chapter, we have employed the virtual element method to solve the coupled nonlocal parabolic equation. First, we prove optimal order convergence for the semi-discrete virtual element formulation with respect to L^2 and H^1 norms. For defining a fully discrete scheme, we use the backward Euler method to discretise the time derivative, and the virtual element method is used for the spatial discretisation. The presence of the nonlocal diffusive coefficients reduces the sparsity of the Jacobian of the nonlinear system. This increase the computational and storage complexity, in contrast to the local problem. To alleviate this difficulty, we have extended Gudi's approach within the context of the virtual element method. In the discrete system of equations, we have introduced two more new variables corresponding to the nonlocal functions g_1 and g_2. The explicit definition of the entries of the Jacobian obtained for the modified system of equations reveals that the Jacobian is sparse. We derived the optimal order error estimates in the L^2 and H^1 norms for the fully discrete scheme. To further reduce the computational complexity, a linearized scheme without compromising the rate of convergence in different norms was proposed. Finally, the theoretical results are justified through numerical experiments over arbitrary polygonal meshes.

Chapter 6

Future Work

As an extension of this thesis, we suggest the following topics for further investigation.

1. We can formulate a computable stabilized VEM scheme for the nonlinear convection-diffusion-reaction equation and derive apriori error estimates under suitable norm for other residual based stabilizers.

2. Some symmetric stabilization methods (for example, the Local Projection stabilization method) have been successfully tested in FEM context. We can investigate these symmetric stabilization methods in the VEM framework. One study the effect of these stabilizer in VEM scheme, in sense of both theoretical analysis and numerical experimentation, for convection dominated problems.

3. We can explore the performance of stabilized VEM for nonstationary and nonlinear problems.

4. We can look into the study of VEM for a system of time-dependent nonlinear convection-diffusion-reaction equations that arise in several practical applications. In fact, this topic is currently under our investigation.

5. We can scrutinize the addition of various stabilization method to nonconforming VEM.

6. In practice, the Navier-Stokes equation have isolated solutions, usually mathematically expressed by the notion of branches of nonsingular solutions. We can propose and analyse VEM approximation of branches of nonsingular solutions.

7. For the two-grid method, we can try to derive optimal apriori error estimates in H^1 norm.

8. Nonlocal models arise in many important practical applications. Few model examples are classical Lotka-Volterra prey-predator model with nonlocal diffusion, nonlocal parabolic systems modelling spread of a disease or epidemic, modelling diffusion in heterogeneous environment to cracks and fractures in composites. We shall consider approximating these nonlocal problems on polygonal or polyhedral discretisation using 2D or 3D VEM.

9. Nonlocal models for convection-diffusion problems exist in literature. For the convection dominated case, we shall consider studying stabilized VEM approximation of the nonlocal convection-diffusion equation.

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List of Publications

Published/Accepted works

1. M Arrutselvi and E Natarajan. Virtual element method for nonlinear convection−diffusion−reaction equation on polygonal meshes.
   International Journal of Computer Mathematics., 98(9):1852−1876, 2021.

2. M Arrutselvi and E Natarajan. Virtual element stabilization of convection−diffusion equation with shock capturing.
   IOP Journal of Physics: Conf. Ser., 1850:1−12, 2021.

3. M Arrutselvi and E Natarajan. Virtual element method for nonlinear time−dependent convection−diffusion−reaction equation.
   Computational Mathematics and Modeling., 32:376−386, 2021.

4. M Arrutselvi, D Adak, E Natarajan, S Roy and S Natarajan. Virtual Element Analysis of Nonlocal Coupled Parabolic Problems on polygonal meshes.
   Calcolo., 59, 18(2), 2022.

Under review

5. M Arrutselvi and E Natarajan. A Shock Capturing Virtual Element Method For The Nonlinear Convection-Diffusion-Reaction Equation.

6. M Arrutselvi, E Natarajan and S Natarajan. Virtual element method for quasilinear convection-diffusion-reaction equation.